LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MAY 26, 2025

REAL LIFE GAMES:
HOW GAME THEORY SHAPES HUMAN DECISIONS

PENALTY SHOOTOUTS & MINIMAX

Adrian Haret
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Belgrade, June 20, 1976.

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At 4-3 for Czechoslovakia, the German striker Uli Hoeneß blasts his shot over the bar.

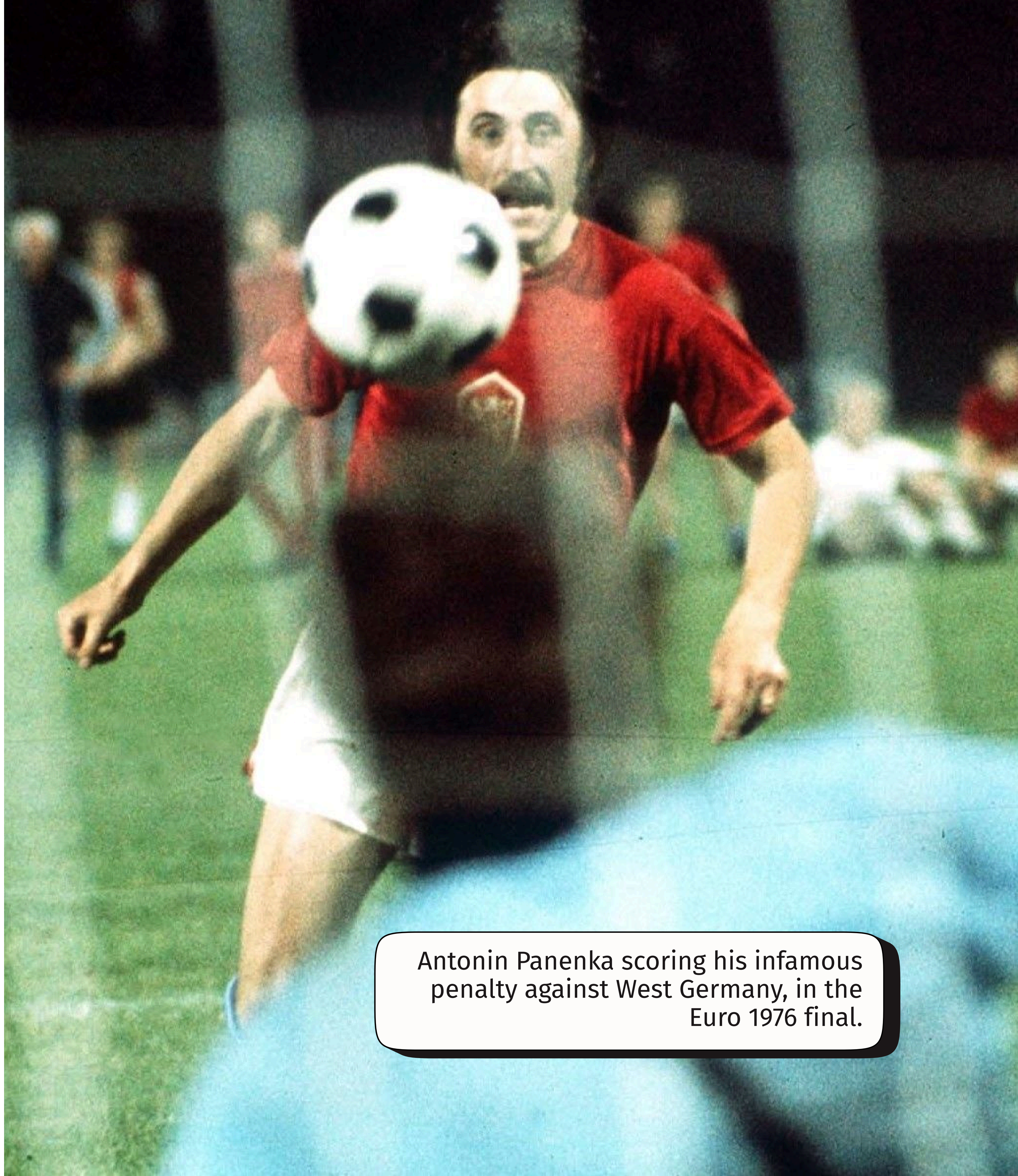
Czechoslovakia can seal the win with a goal. Antonin Panenka steps up…


PANENKA SCORES A PANENKA

Penalty shootouts are ideal objects of study for game theorists.

PENALTY SHOOTOUTS

Clear rules, immediate outcomes.



Antonin Panenka scoring his infamous penalty against West Germany, in the Euro 1976 final.

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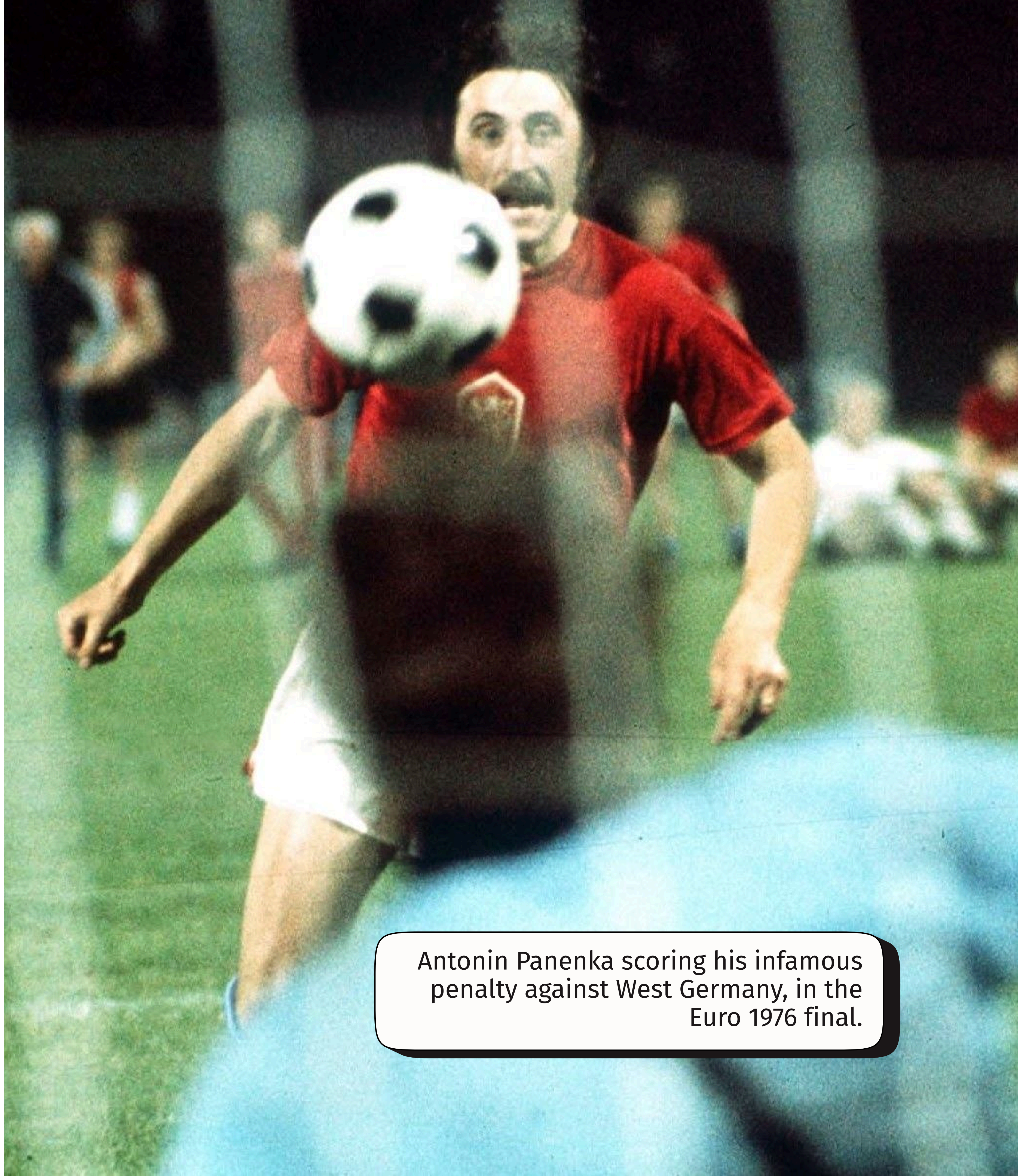

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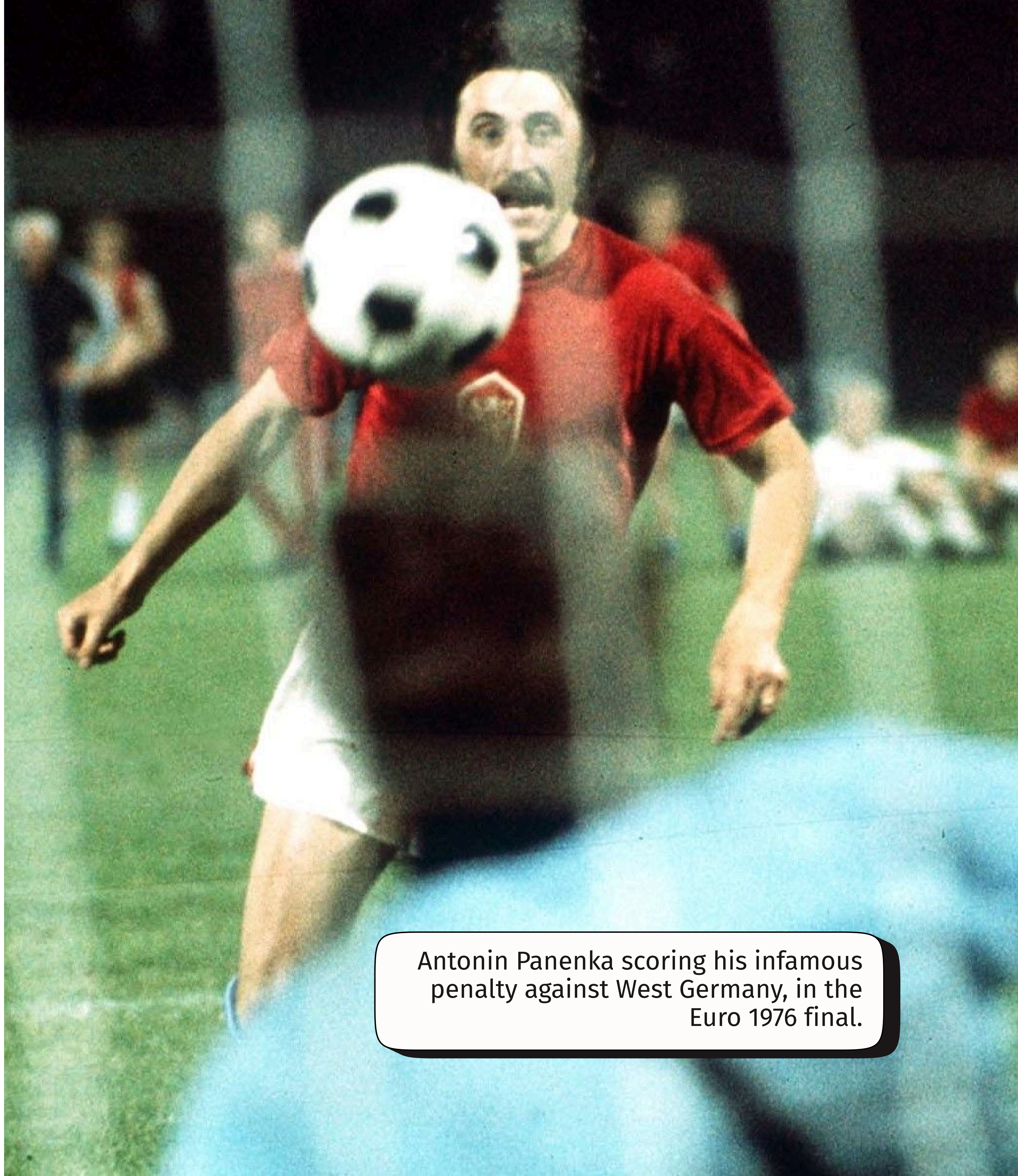

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PENALTY SHOOTOUTS

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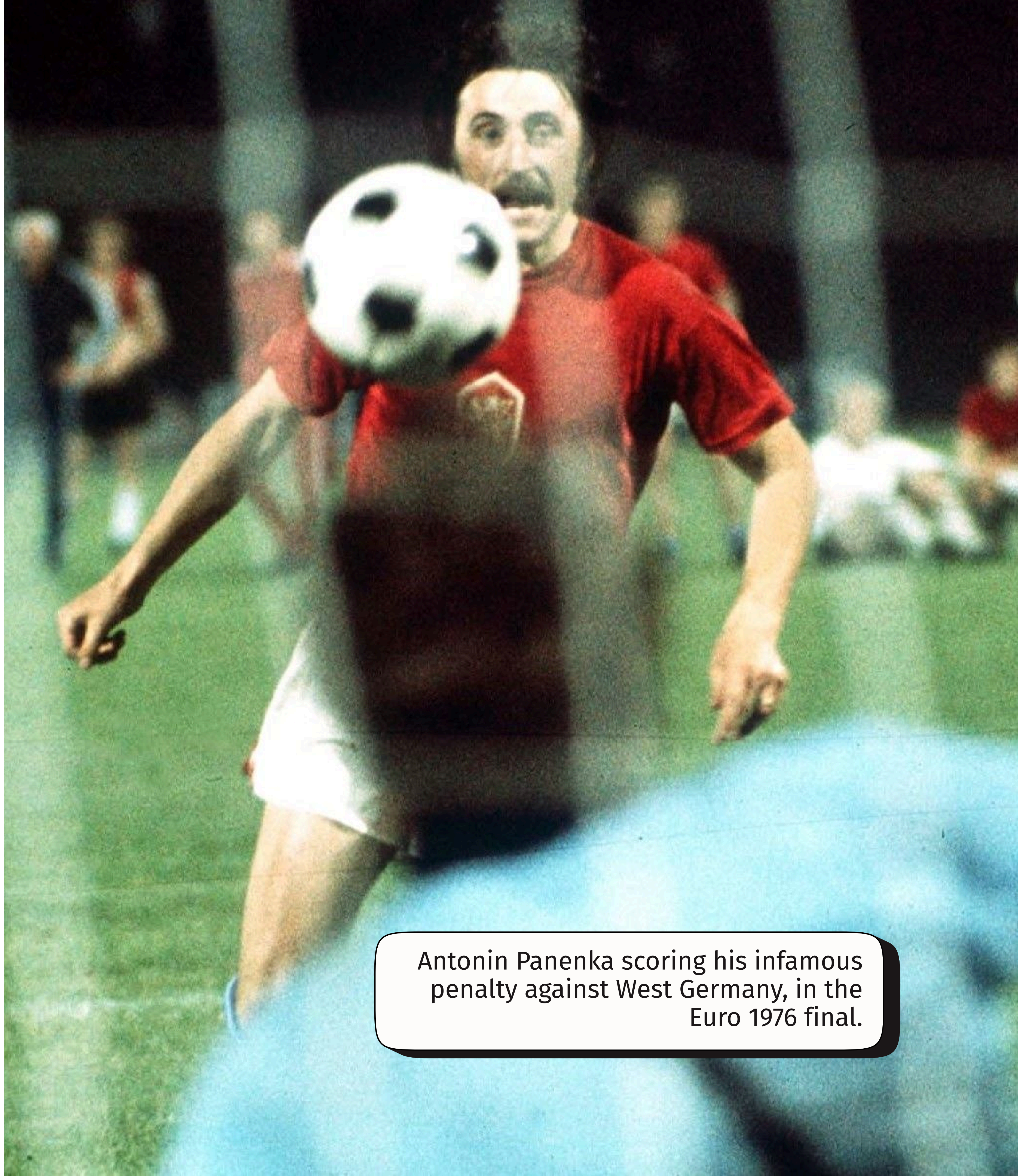
A lot of data available.

Two players involved.

Because it’s so fast, decisions have to be taken simultaneously.



Antonin Panenka scoring his infamous penalty against West Germany, in the Euro 1976 final.

Penalty Shootouts

PERFECT ACCURACY

The game is played between the *Kicker* and the *Goalkeeper*.

The Kicker chooses a direction to shoot in: left (L) or right (R).

The Goalkeeper chooses a direction to dive towards: left (L) or right (R).*

With perfect accuracy on both sides, the Goalkeeper makes a save when matching the direction of the Kicker's shot.

*Everything is from the pov of the Goalkeeper.

payoffs

		GOALKEEPER	
		L (1/2)	R (1/2)
KICKER	L (1/2)	0, 1	1, 0
	R (1/2)	1, 0	0, 1

Pareto optimal strategies

pure Nash equilibria

mixed Nash equilibria

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KICKER \ GOALKEEPER	L (1/2)	R (1/2)
L (1/2)	0, 1	1, 0
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Pareto optimal strategies

all

pure Nash equilibria

none

mixed Nash equilibria

$$s^* = \left((1/2, 1/2), (1/2, 1/2)\right)$$

Suppose, now, that the kicker ocasionally misses when aiming right.

Penalty Shootouts

KICKER 75% ACCURATE TO THE RIGHT

The game is played between the *Kicker* and the *Goalkeeper*.

The Kicker chooses a direction to shoot in: left (L) or right (R).

The Goalkeeper chooses a direction to dive towards: left (L) or right (R).*

The Kicker is accurate 75% of the time when kicking Right.

The expected number of goals scored (and saved) feeds into the payoffs.

*Everything is from the pov of the Goalie.

payoffs

		GOALKEEPER	
		L (q)	R ($1-q$)
KICKER	L (p)	0, 1	1, 0
	R ($1-p$)	$\frac{3}{4}, \frac{1}{4}$	0, 1

Pareto optimal strategies

all

pure Nash equilibria

none

mixed Nash equilibria

The Kicker and Goalkeeper play mixed strategies $s_{\mathsf{K}} = (p, 1-p)$ and $s_{\mathsf{G}} = (q, 1-q)$, respectively.

payoffs

		GOALKEEPER	
		L (4/7)	R (3/7)
KICKER	L (3/7)	$0,\ 1$	$1,\ 0$
	R (4/7)	$\frac{3}{4}, \frac{1}{4}$	$0,\ 1$

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To get the mixed Nash equilibrium, we find the values of p and q that make the Kicker and the Goalkeeper indifferent between their actions:

$$\mathbb{E}\Big[u_{\mathsf{K}}(\mathsf{L}, s_{\mathsf{G}})\Big] = \mathbb{E}\Big[u_{\mathsf{K}}(\mathsf{R}, s_{\mathsf{G}})\Big] \text{ iff } 0 \cdot q + 1 \cdot (1-q) = \frac{3}{4} \cdot q + 0 \cdot (1-q)$$

payoffs

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KICKER	L (3/7)	$0,\ 1$	$1,\ 0$
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$$\text{iff } q = \frac{4}{7}.$$

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KICKER	L (4/7)	R (3/7)
L (3/7)	$0,\ 1$	$1,\ 0$
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&\text{ iff } p = \frac{3}{7}.
\end{aligned}$$

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Interestingly, the Kicker now shoots to their weak side (right) more often!

payoffs

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Interestingly, the Kicker now shoots to their weak side (right) more often!

What's going on here?

payoffs

GOALKEEPER

KICKER		L (4/7)	R (3/7)
	L (3/7)	$0,\ 1$	$1,\ 0$
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The goalkeeper adjusts their *own* strategy by diving right less often.

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Quite subtle.

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Quite subtle. Does it hold up in practice?

IGNACIO PALACIOS-HUERTA

We collected data about 9,017 penalty kicks during the period September 1995 - June 2012.

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Thus, shooting Right means shooting in the player's 'natural' direction.

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Aggregating success rates gives us the following numbers.

Write x, y, z, t, for the various average success rates of the Kicker (see payoffs on the right).

payoffs

		GOALKEEPER L (q)	GOALKEEPER R (1-q)
KICKER	L (p)	x, $1 - x$	y, $1 - y$
KICKER	R (1-p)	z, $1 - z$	t, $1 - t$

Pareto optimal strategies

all

pure Nash equilibria

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mixed Nash equilibria

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Statistics give us the average numbers displayed (as percentages).

payoffs

		GOALKEEPER	
		L (q)	R (1-q)
KICKER	L (p)	$59.11,\ 40.89$	$94.1,\ 5.9$
	R (1-p)	$93.1,\ 6.9$	$71.22,\ 28.78$

Pareto optimal strategies

all

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Write x, y, z, t, for the various average success rates of the Kicker (see payoffs on the right).

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This gives us a very specific prediction, as the mixed Nash equilibrium.

payoffs

GOALKEEPER

KICKER	L (q)	R (1-q)
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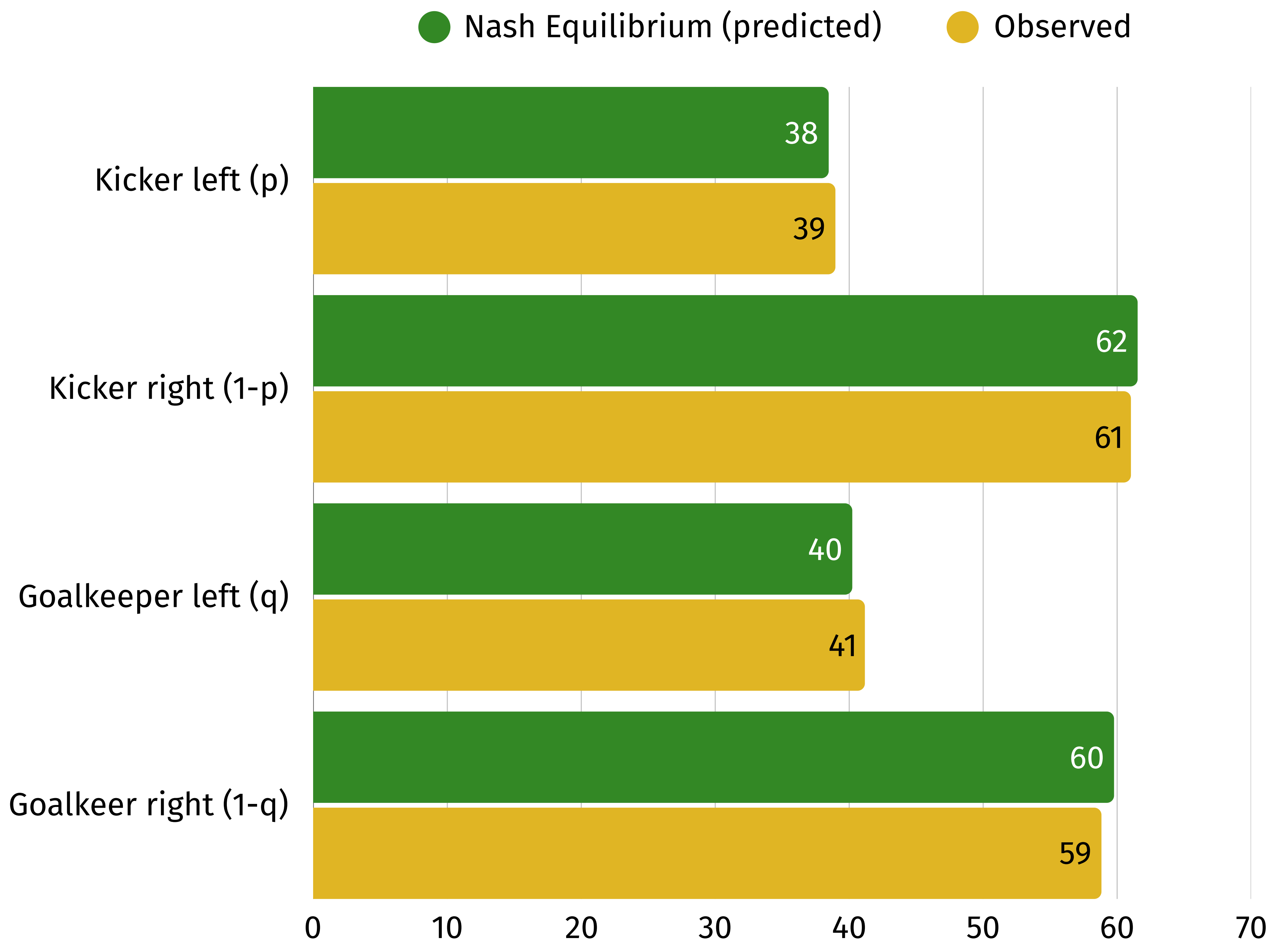
mixed Nash equilibria

$$s^* = \big((38.47\%, 61.53\%), (40.23\%, 59.77\%)\big)$$

Ok, so what do we actually see?

OBSERVED BEHAVIOR

On average, professional players stay very close to the Nash equilibrium!



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This is, at the very least, encouraging for the model.

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Can we then say that players randomize as required by a mixed Nash equilibrium?

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For this to happen, players' choices must be independent draws from a random process.

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For this to happen, players' choices must be independent draws from a random process.

They should not depend on one's own previous play, on the opponent's previous play, on their interaction, or on any other previous actions.

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We can test whether individual players satisfy this using fancy statistical tests.

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Using a player's penalty record, we can test if their behavior is consistent with the equilibrium prediction.

WHAT WE'RE TESTING

Start with a null hypothesis:

H_0 *Players' observed strategies match those predicted by the mixed Nash equilibrium.*

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2 *Players' strategy choices are independent from their own previous choices and from their opponents'.* → Runs test

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A low *p*-value indicates that data strongly disagrees with equilibrium predictions.

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That is, we should reject the hypothesis that the player follow the Nash equilibrium.

Here's what the data tells us.

KICI

Table 1.2. Pearson and Runs Tests

		Proportions		Success Rate		Pearson Tests		Runs Tests		
Name	#Obs	L	R	L	R	Statistic	p-value	r	$\Phi[r-1, s]$	$\Phi[r, s]$
Kickers:										
Mikel Arteta	53	0.433	0.566	0.782	0.833	0.218	0.639	27	0.439	0.551
Alessandro Del Piero	55	0.345	0.654	0.736	0.805	0.344	0.557	24	0.237	0.339
Samuel E'too	62	0.419	0.580	0.769	0.805	0.120	0.728	28	0.165	0.239
Diego Forlán	62	0.419	0.580	0.769	0.805	0.120	0.728	30	0.327	0.427
Steven Gerrard	50	0.340	0.660	0.823	0.909	0.777	0.377	23	0.382	0.507
Thierry Henry	44	0.477	0.522	0.809	0.782	0.048	0.825	19	0.086	0.145
Robbie Keane	42	0.309	0.690	0.769	0.758	1.174	0.278	17	0.184	0.296
Frank Lampard	38	0.236	0.763	0.666	0.793	4.113	0.042**	17	0.791	0.898
Lionel Messi	45	0.377	0.622	1.000	0.928	1.270	0.259	22	0.416	0.544
Alvaro Negredo	45	0.288	0.711	0.769	0.906	1.501	0.220	26	0.986**	0.995
Martín Palermo	55	0.381	0.618	0.714	0.735	0.028	0.865	23	0.098	0.158
Andrea Pirlo	39	0.384	0.615	0.733	0.833	0.566	0.451	20	0.505	0.639
Xabi Prieto	37	0.324	0.675	0.833	0.880	0.151	0.697	16	0.256	0.392
Franc Ribéry	38	0.500	0.500	0.789	0.736	0.145	0.702	25	0.930	0.964
Ronaldinho	46	0.456	0.543	0.952	0.880	0.753	0.385	24	0.460	0.580
Christiano Ronaldo	51	0.372	0.627	0.842	0.718	1.008	0.315	24	0.342	0.458
Roberto Soldado	40	0.400	0.600	0.937	0.750	2.337	0.126	21	0.539	0.667
Francesco Totti	47	0.489	0.510	0.782	0.833	0.195	0.658	20	0.070	0.119
David Villa	52	0.365	0.634	0.631	0.909	5.978	0.014**	18	0.010	0.022**
Zinedine Zidane	61	0.377	0.622	0.782	0.815	0.099	0.752	26	0.126	0.192
All	962	0.386	0.613	0.795	0.822	20.96	0.399			

KICKER STATS

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Christiano Ronaldo	51	0.372	0.627	0.842	0.718	1.008	0.315	24	0.342	0.458
Roberto Soldado	40	0.400	0.600	0.937	0.750	2.337	0.126	21	0.539	0.667
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KICKER STATS

Among top players, only Lampard, Negredo and David Villa fail the test.

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Samuel E'too	62	0.419	0.580	0.769	0.805	0.120	0.728	28	0.165	0.239
Diego Forlán	62	0.419	0.580	0.769	0.805	0.120	0.728	30	0.327	0.427
Steven Gerrard	50	0.340	0.660	0.823	0.909	0.777	0.377	23	0.382	0.507
Thierry Henry	44	0.477	0.522	0.809	0.782	0.048	0.825	19	0.086	0.145
Robbie Keane	42	0.309	0.690	0.769	0.758	1.174	0.278	17	0.184	0.296
Frank Lampard	38	0.236	0.763	0.666	0.793	4.113	0.042**	17	0.791	0.898
Lionel Messi	45	0.377	0.622	1.000	0.928	1.270	0.259	22	0.416	0.544
Alvaro Negredo	45	0.288	0.711	0.769	0.906	1.501	0.220	26	0.986**	0.995
Martín Palermo	55	0.381	0.618	0.714	0.735	0.028	0.865	23	0.098	0.158
Andrea Pirlo	39	0.384	0.615	0.733	0.833	0.566	0.451	20	0.505	0.639
Xabi Prieto	37	0.324	0.675	0.833	0.880	0.151	0.697	16	0.256	0.392
Franc Ribéry	38	0.500	0.500	0.789	0.736	0.145	0.702	25	0.930	0.964
Ronaldinho	46	0.456	0.543	0.952	0.880	0.753	0.385	24	0.460	0.580
Christiano Ronaldo	51	0.372	0.627	0.842	0.718	1.008	0.315	24	0.342	0.458
Roberto Soldado	40	0.400	0.600	0.937	0.750	2.337	0.126	21	0.539	0.667
Francesco Totti	47	0.489	0.510	0.782	0.833	0.195	0.658	20	0.070	0.119
David Villa	52	0.365	0.634	0.631	0.909	5.978	0.014**	18	0.010	0.022**
Zinedine Zidane	61	0.377	0.622	0.782	0.815	0.099	0.752	26	0.126	0.192
All	962	0.386	0.613	0.795	0.822	20.96	0.399			

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KICKER STATS

Among top players, only Lampard, Negredo and David Villa fail the test.

They're a bit too predictable!

Table 1.2. Pearson and Runs Tests

		Proportions		Success Rate		Pearson Tests		Runs Tests		
Name	#Obs	L	R	L	R	Statistic	p-value	r	$\Phi[r-1, s]$	$\Phi[r, s]$
Kickers:										
Mikel Arteta	53	0.433	0.566	0.782	0.833	0.218	0.639	27	0.439	0.551
Alessandro Del Piero	55	0.345	0.654	0.736	0.805	0.344	0.557	24	0.237	0.339
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All	962	0.386	0.613	0.795	0.822	20.96	0.399			

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GOALIE STATS

And among top goalkeepers, van der Sar and Lehmann fail the test.

Goalkeepers:										
Dani Aranzubia	68	0.455	0.544	0.225	0.189	0.138	0.709	29	0.062	0.098
Gianluigi Buffon	71	0.408	0.591	0.241	0.142	1.113	0.291	35	0.420	0.518
Willie Caballero	60	0.350	0.650	0.095	0.230	1.674	0.195	29	0.522	0.634
Iker Casillas	69	0.347	0.652	0.250	0.088	3.278	0.070*	32	0.414	0.520
Petr Čech	82	0.414	0.585	0.235	0.187	0.276	0.590	38	0.224	0.298
Júlio César	68	0.308	0.691	0.238	0.106	2.007	0.156	34	0.840	0.900
Morgan De Sanctis	62	0.435	0.564	0.148	0.342	3.018	0.082*	34	0.700	0.783
Tim Howard	67	0.402	0.597	0.222	0.225	0.000	0.978	30	0.169	0.241
Bodo Illgner	68	0.352	0.647	0.250	0.272	0.041	0.839	33	0.547	0.650
Gorka Iraizoz	73	0.424	0.575	0.129	0.142	0.028	0.865	32	0.106	0.157
David James	69	0.391	0.608	0.185	0.238	0.270	0.603	40	0.924	0.954
Oliver Kahn	58	0.379	0.620	0.227	0.138	0.747	0.387	33	0.881	0.928
Andreas Kopke	70	0.428	0.571	0.233	0.150	0.787	0.374	31	0.119	0.175
Jens Lehman	72	0.444	0.555	0.218	0.225	0.004	0.949	28	0.014	0.026*
Andrés Palop	66	0.439	0.560	0.206	0.297	0.694	0.404	34	0.498	0.597
Pepe Reina	55	0.418	0.581	0.173	0.187	0.016	0.897	31	0.778	0.852
Mark Schwarzer	55	0.381	0.618	0.238	0.264	0.048	0.825	31	0.846	0.904
Stefano Sorrentino	48	0.458	0.541	0.136	0.269	1.275	0.258	27	0.687	0.783
Víctor Valdes	71	0.394	0.605	0.214	0.232	0.032	0.857	32	0.196	0.272
Edwin van der Sar	80	0.412	0.587	0.121	0.148	0.125	0.722	26	0.000	0.001**
All	1332	0.402	0.597	0.199	0.198	15.58	0.742			

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IGNACIO PALACIOS-HUERTA

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All the Chelsea players shot to his left.

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Data showed that van der Sar, the Man Utd keeper, had a slight tendency to dive to the right against right-footed players.

All the Chelsea players shot to his left.

Then Anelka stepped up to the plate...

https://youtu.be/z-QliFMvpqI?t=221

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JOHN VON NEUMANN

These are all applications of the Minimax theorem for zero-sum games.

A game is zero-sum when one player’s win is the other’s loss.

DEFINITION

A two-player game is *zero-sum* if payoffs add up to zero in every outcome. Specifically, if Player 1 plays action x and Player 2 plays action y, then:

$$u_1(x, y) + u_2(x, y) = 0.$$

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$$u_1(x, y) + u_2(x, y) = 0.$$

In other words, $u_1(x, y) = -u_2(x, y)$.

Examples?

ROCK-PAPER-SCISSORS

Paper beats Rock, Scissors beats Paper, Rock beats Scissors.

And same-same is a tie.

payoffs

	Rock (1/3)	Paper (1/3)	Scissors (1/3)
Rock (1/3)	0, 0	-1, 1	1, -1
Paper (1/3)	1, -1	0, 0	-1, 1
Scissors (1/3)	-1, 1	1, -1	0, 0

pure Nash equilibria

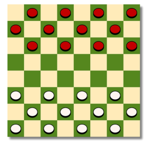
none

mixed Nash equilibria

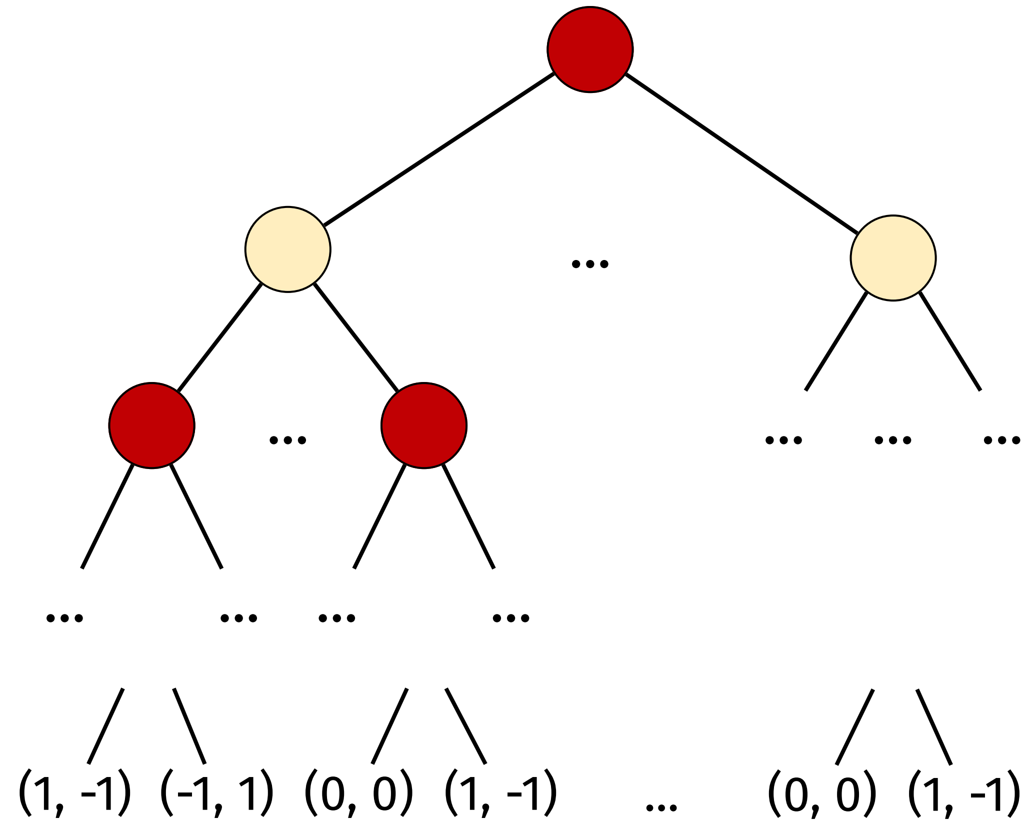
$$\boldsymbol{s}^* = \left((1/3, 1/3, 1/3), (1/3, 1/3, 1/3) \right)$$

CHECKERS

Winner gets 1, loser gets -1. In a tie, each gets 0.



payoffs



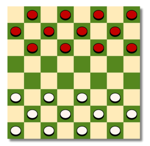
pure Nash equilibria

mixed Nash equilibria

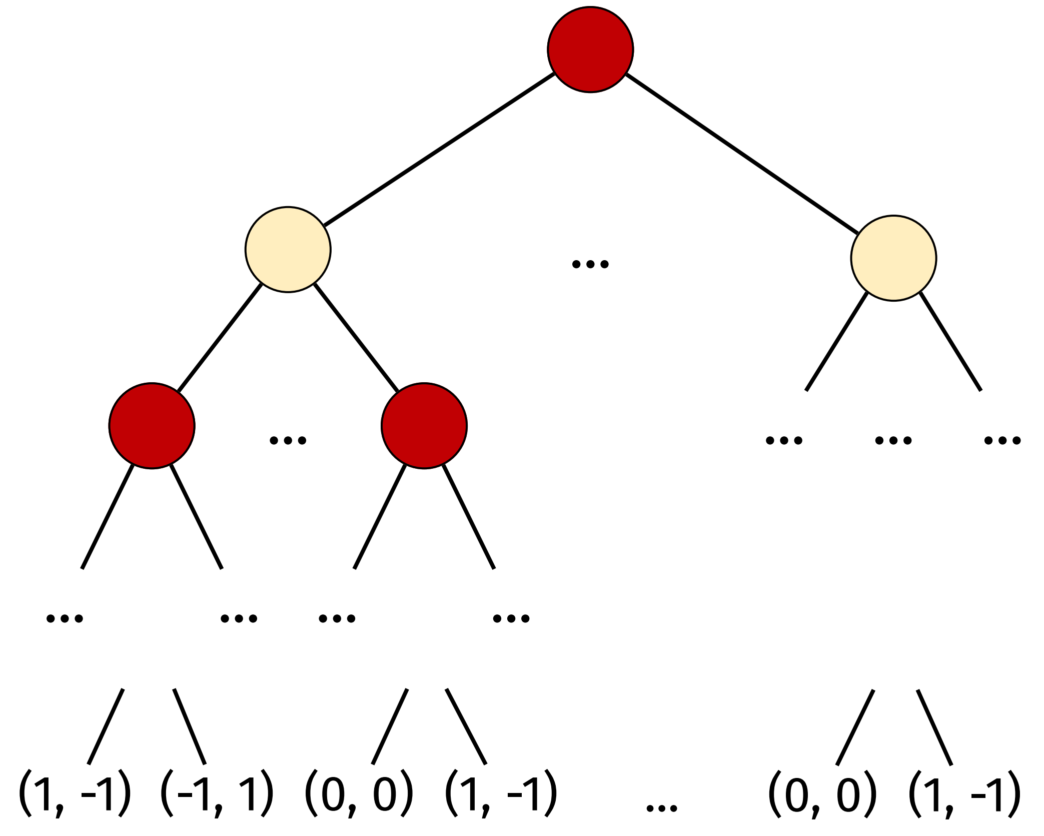
CHECKERS

Winner gets 1, loser gets -1. In a tie, each gets 0.

With 24 pieces there are 500,995,484,682,338,672,639 (~5×10^{20}) possible positions of the board.



payoffs



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An algorithm was found ensuring that, regardless of what the other player does, you do not lose.

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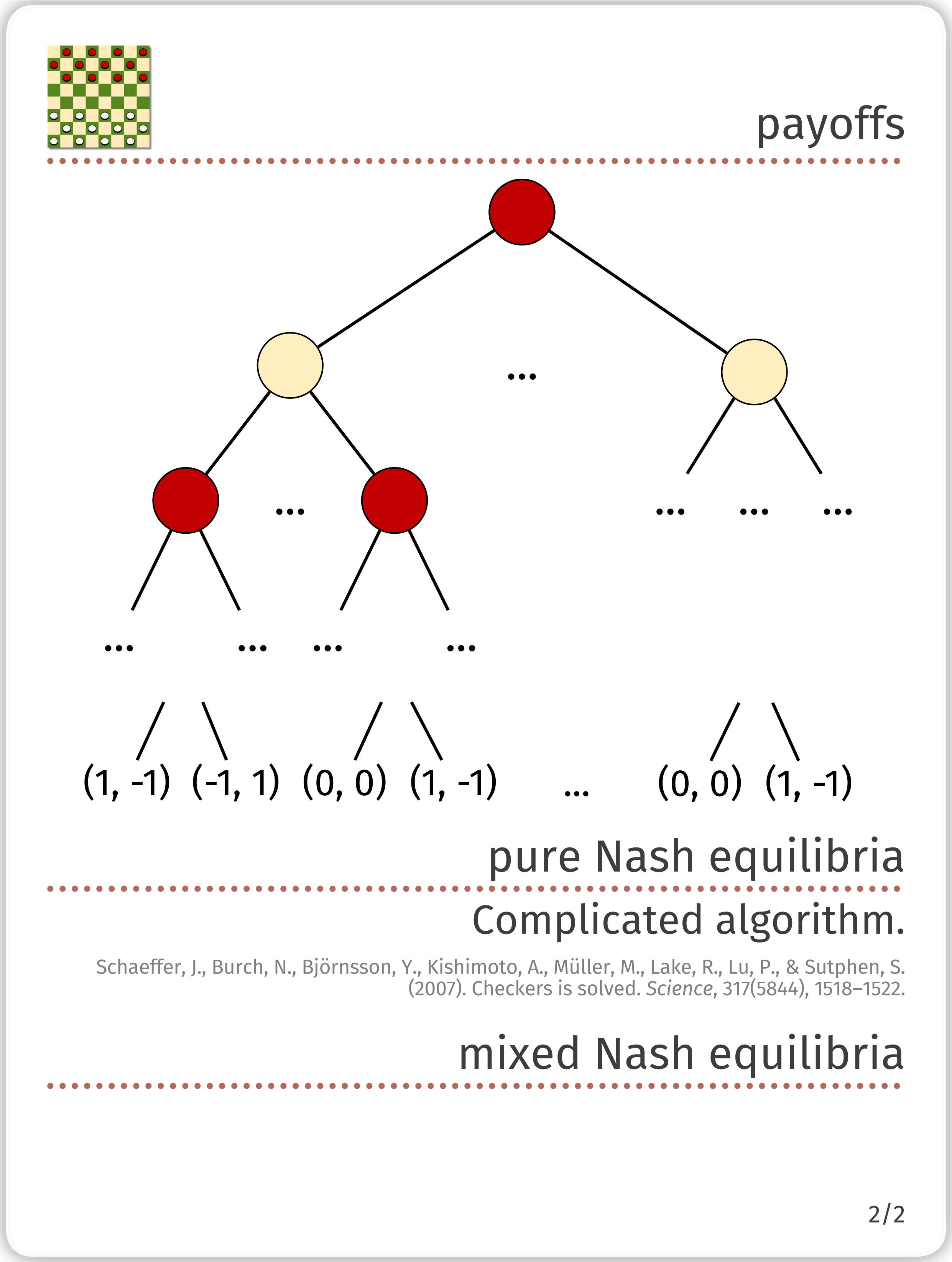

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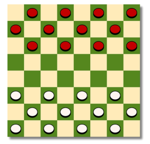
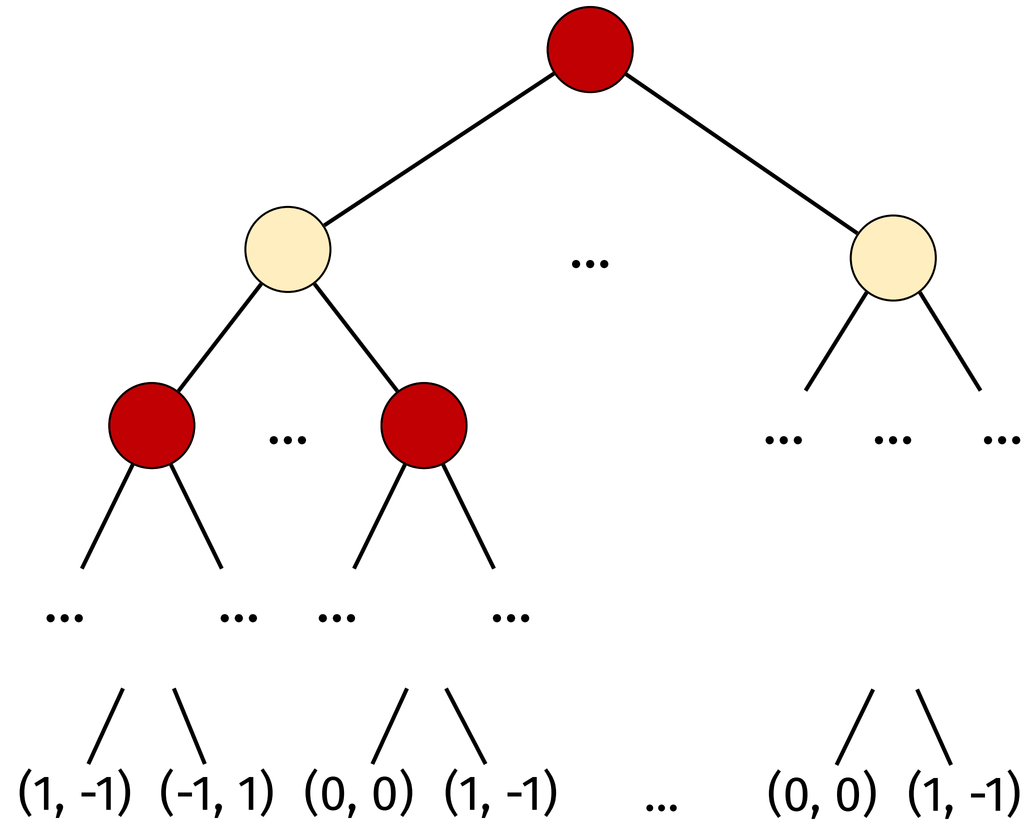
So when both players play perfectly, the game results in a draw.



payoffs



pure Nash equilibria

Complicated algorithm.

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mixed Nash equilibria

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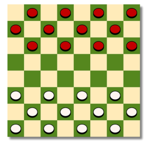
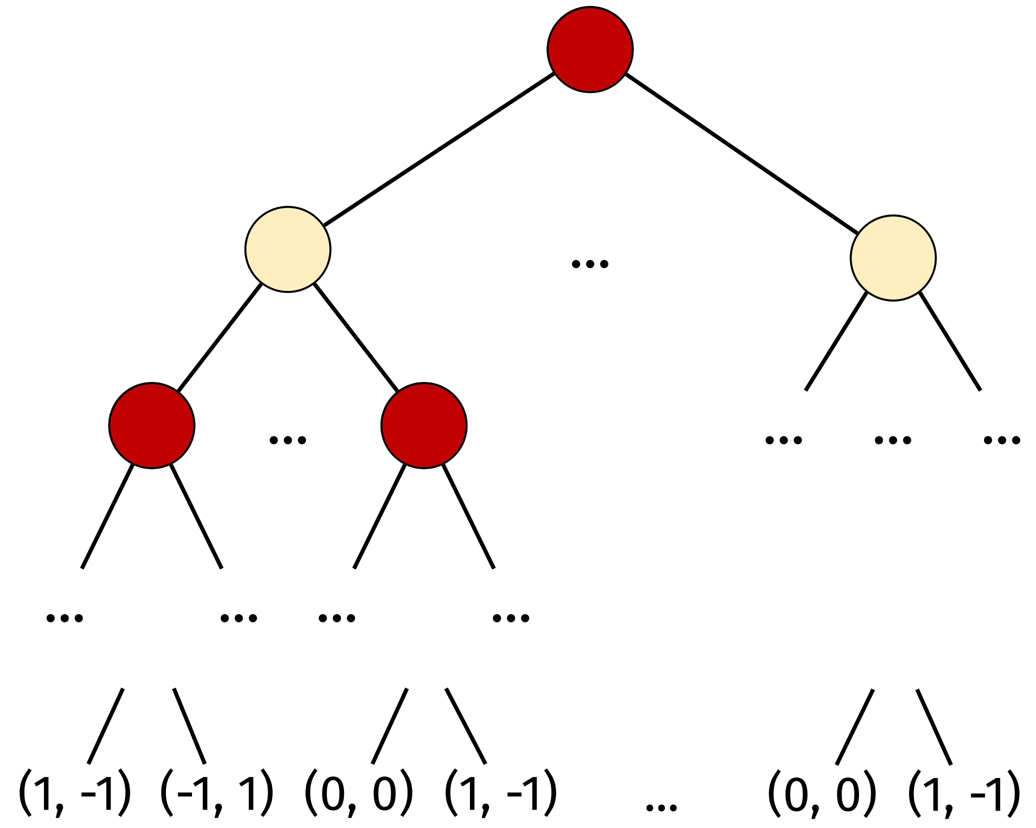
So when both players play perfectly, the game results in a draw.

This is equivalent to an equilibrium in pure strategies.



payoffs



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mixed Nash equilibria

Not needed, by Zermelo’s Theorem.

Zermelo, E. (1912).¨Uber eine Anwendung der Mengenlehre auf die Theorie des Schachspiels

JONATHAN SCHAEFFER

Rather than starting at the opening and moving forward, we worked backwards from end positions.

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Rather than starting at the opening and moving forward, we worked backwards from end positions.

We built an enormous database of endgames and reasoned through the game tree.

We used over 200 computers, on and off, for almost two decades to cover all relevant branches.

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CHESS

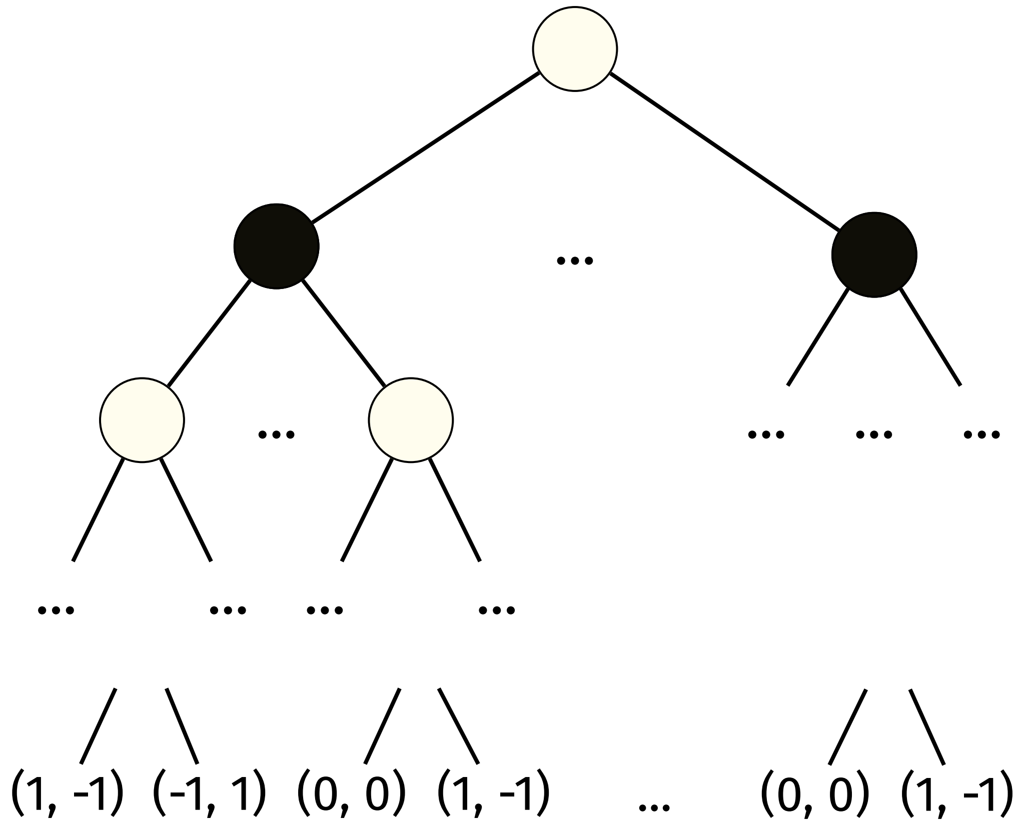
Winner gets 1, loser gets -1. In a tie, each gets 0.

An estimated 10^{43} - 10^{50} legal positions.

Unlike Checkers, Chess is not fully solved.

Based on AI evidence, it is thought that perfect play leads do a draw.

payoffs



pure Nash equilibria

Must exist, but we don’t know what they are and which player they favor.

Zermelo, E. (1912). ̈Uber eine Anwendung der Mengenlehre auf die Theorie des Schachspiels

mixed Nash equilibria

Not needed, by Zermelo’s Theorem.

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2/2

Complementary payoffs means we can focus on only one side of the payoffs.

WRITING ZERO-SUM GAMES

Since Player 2's payoffs are just the opposite of Player 1's, we can leave them out.

payoffs

	A	B
A	3, -3	0, 0
B	2, -2	1, -1

WRITING ZERO-SUM GAMES

Since Player 2's payoffs are just the opposite of Player 1's, we can leave them out.

The numbers in the boxes represent Player 1's payoffs.

payoffs

	A	B
A	3, -3	0, 0
B	2, -2	1, -1

WRITING ZERO-SUM GAMES

Since Player 2's payoffs are just the opposite of Player 1's, we can leave them out.

The numbers in the boxes represent Player 1's payoffs.

Which Player 1 wants to maximize...

payoffs

	A	B
A	3, -3	0, 0
B	2, -2	1, -1

WRITING ZERO-SUM GAMES

Since Player 2's payoffs are just the opposite of Player 1's, we can leave them out.

The numbers in the boxes represent Player 1's payoffs.

Which Player 1 wants to maximize...

... and Player 2 wants to *minimize*.

payoffs

	A	B
A	3, -3	0, 0
B	2, -2	1, -1

WRITING ZERO-SUM GAMES

Since Player 2's payoffs are just the opposite of Player 1's, we can leave them out.

The numbers in the boxes represent Player 1's payoffs.

Which Player 1 wants to maximize...

... and Player 2 wants to *minimize*.

Everything else (e.g., Nash equilibria, Pareto optimal outcomes) stays the same.

payoffs

	A	B
A	3	0
B	2	1

Pareto optimal strategies

all

pure Nash equilibria

$s^* = (\mathsf{B}, \mathsf{B})$

Consider the following way to play a game.

DEFINITION (MINMAXIMIZER)

Assume Player 1 is a *Minmaximizer*, which means they pick the strategy that maximizes their minimum payoff:

$$\max_{s_1} \min_{s_2} u_1\left(s_1, s_2\right).$$

Player 1 is cautious, i.e., picks the strategy that gives them the best worst-case scenario, assuming Player 2 wants to screw them over.

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Player 1 is cautious, i.e., picks the strategy that gives them the best worst-case scenario, assuming Player 2 wants to screw them over.

DEFINITION (MINMAXIMIZER)

Assume Player 2 is a *Maxminimizer*, which means they pick the strategy that minimizes the maximum payoff of Player 1:

$$\min_{s_2} \max_{s_1} u_1\left(s_1, s_2\right).$$

EXAMPLE

Suppose we allow only pure strategies.

Player 1 thinks as follows:

> *If I choose A, the worst I can get is 0.*
>
> *If I choose B, the worst I can get is 1.*
>
> *Getting 1 is better than getting 0.*

The max-min value is:

$$\max\{0, 1\} = 1.$$

payoffs

PLAYER 2

		A	B
PLAYER 1	A	3	0
	B	2	1

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>
> *If I choose B, the worst I can get is 1.*
>
> *Getting 1 is better than getting 0.*

The max-min value is:

$$\max\{0, 1\} = 1.$$

Player 2 thinks as follows:

> *If I choose A, the best Player 1 can do is 3.*
>
> *If I choose B, the best Player 1 can do is 1.*
>
> *Player 1's minimal payoff is 1.*

The min-max value is:

$$\min\{3, 1\} = 1.$$

payoffs

		PLAYER 2 A	PLAYER 2 B
PLAYER 1	A	3	0
PLAYER 1	B	2	1

The max-min and min-max values coincide in this case. They don’t need to.

ANOTHER EXAMPLE

Suppose we still allow only pure strategies.

The max-min value for Player 1 is:

$$\max\{-1, -1\} = -1.$$

The min-max value for Player 2 is:

$$\min\{1, 1\} = 1.$$

payoffs

		PLAYER 2	
		Heads	Tails
PLAYER 1	Heads	1	-1
	Tails	-1	1

MINIMAXING W/ MIXED STRATEGIES

Players 1 and 2 play mixed strategies $s_1 = (p, 1-p)$ and $s_2 = (q, 1-q)$, respectively.

And they want to maximize (minimize, respectively) the expected payoff:

payoffs

PLAYER 2

PLAYER 1	H (q)	T (1-q)
H (p)	1	-1
T (1-p)	-1	1

MINIMAXING W/ MIXED STRATEGIES

Players 1 and 2 play mixed strategies $s_1 = (p, 1-p)$ and $s_2 = (q, 1-q)$, respectively.

And they want to maximize (minimize, respectively) the expected payoff:

$$\mathbb{E}\Big[u_1(s_1, s_2)\Big] = \mathbb{E}\Big[u_1(\mathsf{H}, s_2)\Big] \cdot p + \mathbb{E}\Big[u_1(\mathsf{T}, s_2)\Big] \cdot (1-p)$$

payoffs

PLAYER 2

PLAYER 1	H (q)	T (1-q)
H (p)	1	-1
T (1-p)	-1	1

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Players 1 and 2 play mixed strategies $s_1 = (p, 1-p)$ and $s_2 = (q, 1-q)$, respectively.

And they want to maximize (minimize, respectively) the expected payoff:

$$\begin{aligned}\mathbb{E}\Big[u_1(s_1, s_2)\Big] &= \mathbb{E}\Big[u_1(\mathsf{H}, s_2)\Big] \cdot p + \mathbb{E}\Big[u_1(\mathsf{T}, s_2)\Big] \cdot (1-p) \\ &= \Big(u_1(\mathsf{H},\mathsf{H}) \cdot q + u_1(\mathsf{H},\mathsf{T}) \cdot (1-q)\Big) \cdot p + \\ &\quad \Big(u_1(\mathsf{T},\mathsf{H}) \cdot q + u_1(\mathsf{T},\mathsf{T}) \cdot (1-q)\Big) \cdot (1-p)\end{aligned}$$

payoffs

		PLAYER 2	
		H (*q*)	T (1-*q*)
PLAYER 1	H (*p*)	1	-1
	T (1-*p*)	-1	1

MINIMAXING W/ MIXED STRATEGIES

Players 1 and 2 play mixed strategies $s_1 = (p, 1-p)$ and $s_2 = (q, 1-q)$, respectively.

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\mathbb{E}\Big[u_1\big(s_1, s_2\big)\Big] &= \mathbb{E}\Big[u_1\big(\mathsf{H}, s_2\big)\Big]\cdot p + \mathbb{E}\Big[u_1\big(\mathsf{T}, s_2\big)\Big]\cdot(1-p) \\
&= \Big(u_1\big(\mathsf{H},\mathsf{H}\big)\cdot q + u_1\big(\mathsf{H},\mathsf{T}\big)\cdot(1-q)\Big)\cdot p + \\
&\quad\ \Big(u_1\big(\mathsf{T},\mathsf{H}\big)\cdot q + u_1\big(\mathsf{T},\mathsf{T}\big)\cdot(1-q)\Big)\cdot(1-p) \\
&= u_1\big(\mathsf{H},\mathsf{H}\big)\cdot p\cdot q + u_1\big(\mathsf{H},\mathsf{T}\big)\cdot p\cdot(1-q) + \\
&\quad\ u_1\big(\mathsf{T},\mathsf{H}\big)\cdot(1-p)\cdot q + u_1\big(\mathsf{T},\mathsf{T}\big)\cdot(1-p)\cdot(1-q) \\
&= 4pq - 2p - 2q + 1.
\end{aligned}$$

payoffs

PLAYER 2

PLAYER 1		H (q)	T (1-q)
	H (p)	1	-1
	T (1-p)	-1	1

MINIMAXING W/ MIXED STRATEGIES

Think of the expected utility as a function of p and q:

$$f(p, q) = 4pq - 2p - 2q + 1.$$

We want to find $\max_p \min_q f(p, q)$.

payoffs

PLAYER 1 \ PLAYER 2	H (q)	T (1-q)
H (p)	1	-1
T (1-p)	-1	1

MINIMAXING W/ MIXED STRATEGIES

Think of the expected utility as a function of p and q:

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Player 2 wants to minimize $f(p, q)$ by choosing q. To find the value for q, take the partial derivative of f with respect to q:

$$\frac{\partial f}{\partial q} = 4p - 2.$$

payoffs

		PLAYER 2	
		H (q)	T (1-q)
PLAYER 1	H (p)	1	-1
	T (1-p)	-1	1

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$$\frac{\partial f}{\partial q} = 4p - 2.$$

The sign of the partial derivative tells us whether f is increasing or decreasing with respect to q. If $4p - 2 < 0$, f is decreasing and Player 2 sets $q = 1$. If $4p - 2 > 0$, f is increasing and Player 2 sets $q = 0$. If $4p - 2 = 0$, f is constant at $f(p, 1/2) = 0$.

payoffs

		PLAYER 2	
		H (q)	T (1-q)
PLAYER 1	H (p)	1	-1
	T (1-p)	-1	1

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So Player 1's worst-case payoff is:

$$\min_q f(p, q) = \begin{cases} 2p - 1, & \text{if } 0 \leq p < 1/2, \\ 0, & \text{if } p = 1/2, \\ -2p + 1, & \text{if } 1/2 < p \leq 1. \end{cases}$$

payoffs

		PLAYER 2	
		H (q)	T (1-q)
PLAYER 1	H (p)	1	-1
	T (1-p)	-1	1

MINIMAXING W/ MIXED STRATEGIES

Think of the expected utility as a function of p and q:

$$f(p,q) = 4pq - 2p - 2q + 1.$$

We want to find $\max_p \min_q f(p,q)$.

Player 2 wants to minimize $f(p,q)$ by choosing q. To find the value for q, take the partial derivative of f with respect to q:

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So Player 1's worst-case payoff is:

$$\min_q f(p,q) = \begin{cases} 2p - 1, & \text{if } 0 \leq p < {}^1\!/_2, \\ 0, & \text{if } p = {}^1\!/_2, \\ -2p + 1, & \text{if } {}^1\!/_2 < p \leq 1. \end{cases}$$

Player 1 wants to maximize this worst-case payoff, which in this case happens at $p^* = {}^1\!/_2$.

payoffs

PLAYER 2

PLAYER 1	H (q)	T (1-q)
H (p)	1	-1
T (1-p)	-1	1

MINIMAXING W/ MIXED STRATEGIES

The symmetric calculation shows that Player 2's strategy that minimizes Player 1's best-case expected payoff is:

$$q^* = {}^1/_2.$$

payoffs

PLAYER 2

PLAYER 1		H (q)	T (1-q)
	H (p)	1	-1
	T (1-p)	-1	1

MINIMAXING W/ MIXED STRATEGIES

The symmetric calculation shows that Player 2's strategy that minimizes Player 1's best-case expected payoff is:

$$q^* = {}^1\!/_2.$$

Note that in this case:

$$\max_p \min_q f(p,q) = \min_q \max_p f(p,q) = 0.$$

payoffs

		PLAYER 2	
		H (q)	T (1-q)
PLAYER 1	H (p)	1	-1
	T (1-p)	-1	1

Remarkably, this generalizes!

THEOREM (VON NEUMANN, 1928)

In any finite two-player zero-sum game, the maximum value a player can guarantee by choosing a strategy (regardless of the opponent's strategy) is equal to the minimum value the opponent can force upon them:

$$\max_{s_1} \min_{s_2} \mathbb{E}\Big[u_1\big(s_1, s_2\big)\Big] = \min_{s_2} \max_{s_1} \mathbb{E}\Big[u_1\big(s_1, s_2\big)\Big].$$

von Neumann, J. (1928). Zur Theorie der Gesellschaftsspiele. *Mathematische Annalen* 100, 295–320.

The common value for the best worst-case and the worst best-case is also called the *value* of the game.

JOHN VON NEUMANN

I thought there was nothing worth publishing until the Minimax Theorem was proved.

THEOREM

The maxmin and minmax strategies form a Nash equilibrium.