LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

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REAL LIFE GAMES:
HOW GAME THEORY SHAPES HUMAN DECISIONS

GAME THEORY

MIXED NASH EQUILIBRIA

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Let’s play a new, exciting game!

Let’s play a new, exciting game! Can you beat Adrian at Rock-Paper-Scissors?

Pure Nash equilibria always exist.

Pure Nash equilibria always exist.
Except when they don't.

Matching Pennies

Two players have a penny each.

They decide on a face and reveal it at the same time.

If the faces match, player 1 wins $1, player 2 loses $1.

If the faces do not match, player 2 wins $1, player 1 loses $1.

payoffs

	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

Pareto optimal strategies

all

pure Nash equilibria

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payoffs

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Pareto optimal strategies

all

pure Nash equilibria

none

There is, however, a different way to play this game.

JOHN NASH

Sometimes the best thing to do is to flip a coin.

MIXED STRATEGIES

DEFINITION

A *mixed strategy* s_i for player i is a probability distribution over i's actions, written $s_i = (p_1, \ldots, p_j, \ldots)$, where p_j is the probability with which player i plays action j.

Note that it needs to hold that $\sum_j p_j = 1$ and $p_j \geq 0$.

MIXED STRATEGIES: EXAMPLE

If Player 1 plays $s_1 = (0.9, 0.1)$, that means they play Heads with probability 0.9 and Tails with probability 0.1.

Note that the pure strategies we've been dealing with so far are special cases of mixed strategies, in which one action is played with probability 1 and the rest with probability 0.

payoffs

	Heads	Tails
Heads (0.9)	1, -1	-1, 1
Tails (0.1)	-1, 1	1, -1

Pareto optimal strategies

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2/2

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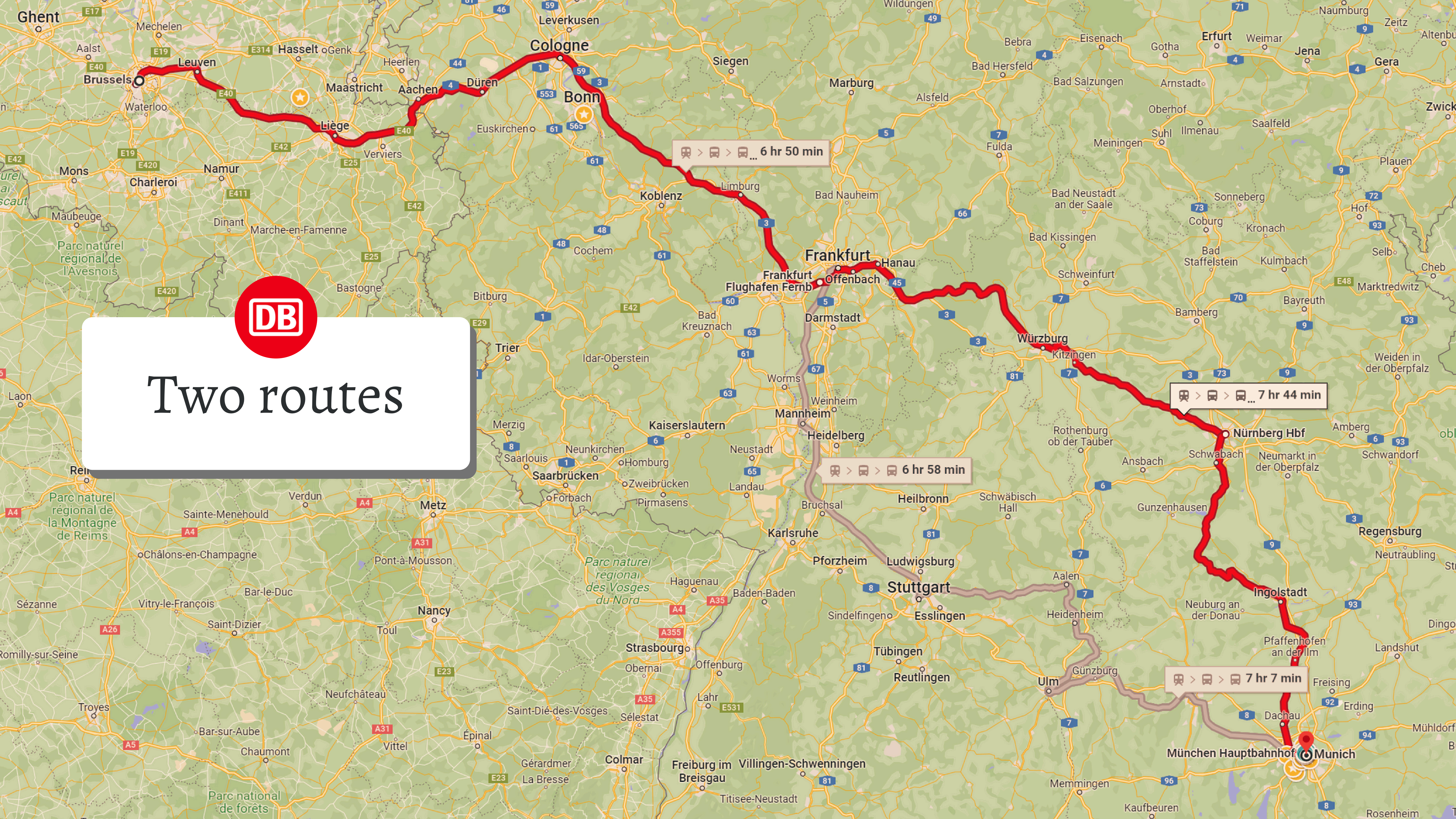
ADRIAN

I need to get from Brussels to Munich.



Two routes
6 hr 50 min
7 hr 44 min
6 hr 58 min
7 hr 7 min
Brussels
Cologne
Frankfurt
Nürnberg Hbf
Stuttgart
München Hauptbahnhof
Munich

EXPECTED UTILITY: EXAMPLE

My utility is determined by the arrival time.

Option 1

Brussels - Frankfurt - München

23:00

Option 2

Brussels - Köln - München

23:20

EXPECTED UTILITY: EXAMPLE

My utility is determined by the arrival time.

Option 1
Brussels - Frankfurt - München

0

Option 2
Brussels - Köln - München

-20

EXPECTED UTILITY: EXAMPLE

My utility is determined by the arrival time.

Which option is best?

Option 1
Brussels - Frankfurt - München

0

Option 2
Brussels - Köln - München

-20

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My utility is determined by the arrival time.

Which option is best?

Option 1
Brussels - Frankfurt - München

0

Option 2
Brussels - Köln - München

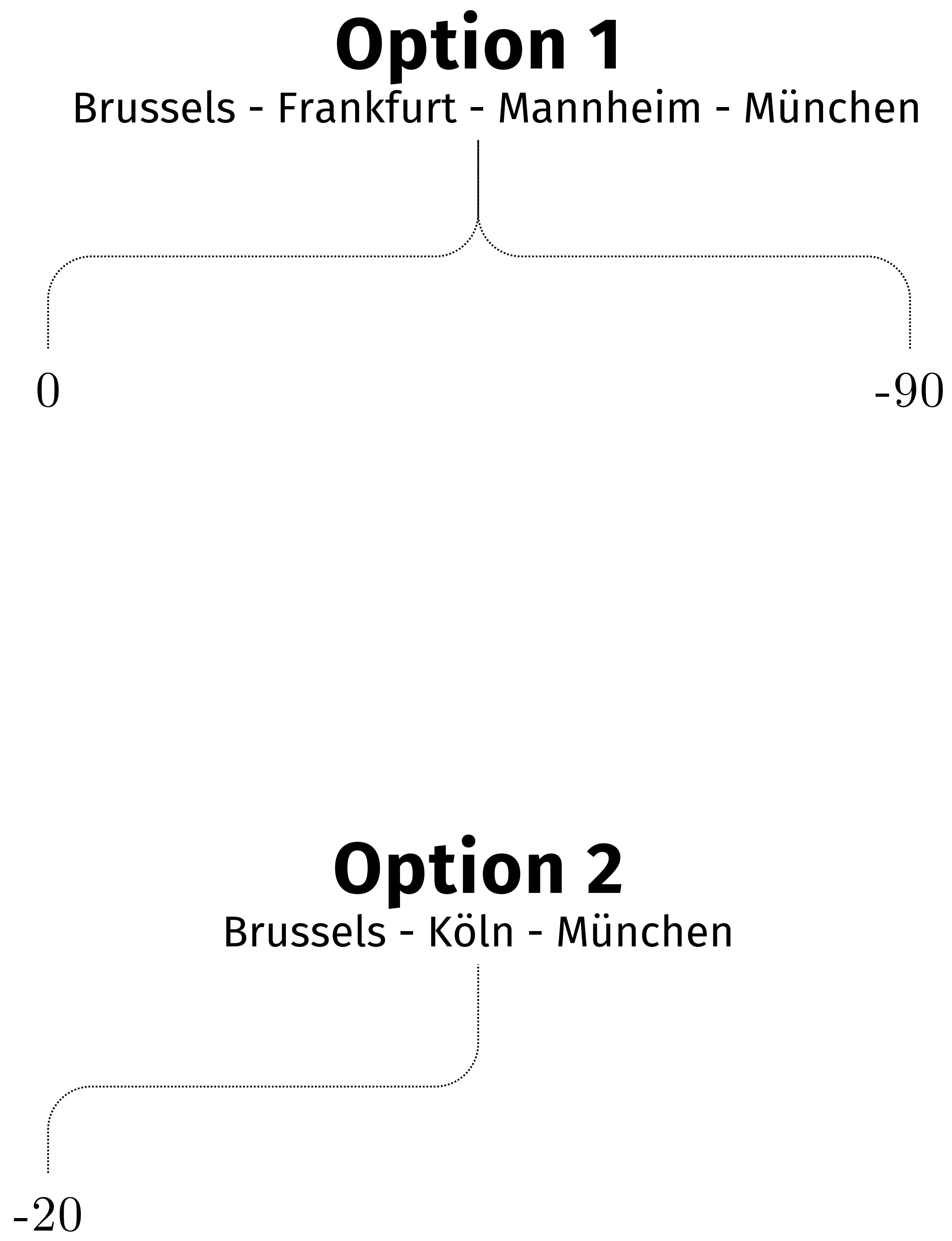
-20

EXPECTED UTILITY: EXAMPLE

My utility is determined by the arrival time.

Which option is best?

But with the first option I might miss the Frankfurt connection, meaning and will get home even later.



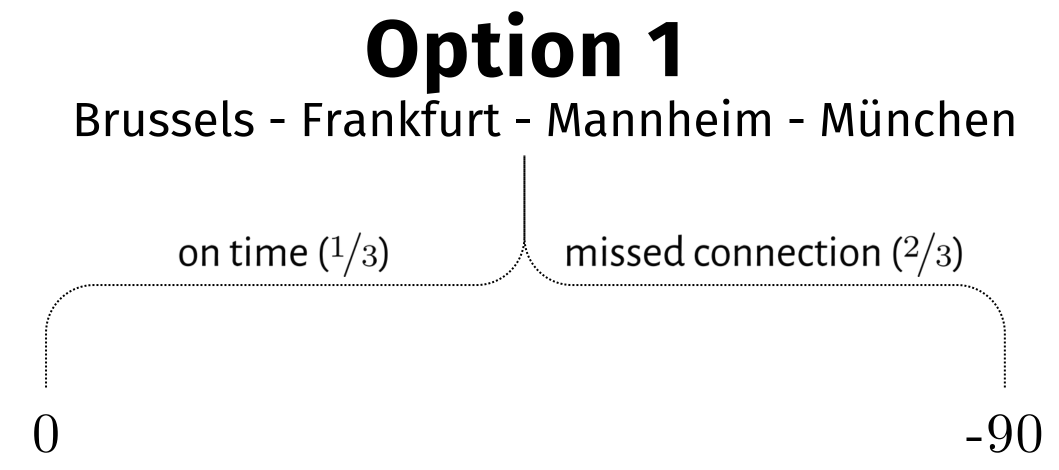
EXPECTED UTILITY: EXAMPLE

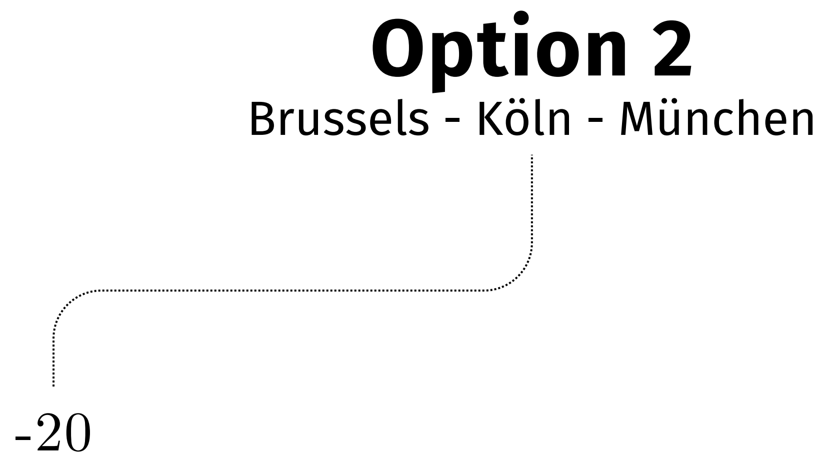
My utility is determined by the arrival time.

Which option is best?

But with the first option I might miss the Frankfurt connection, meaning and will get home even later.

This is very likely to happen... So how should we think of this possibility?





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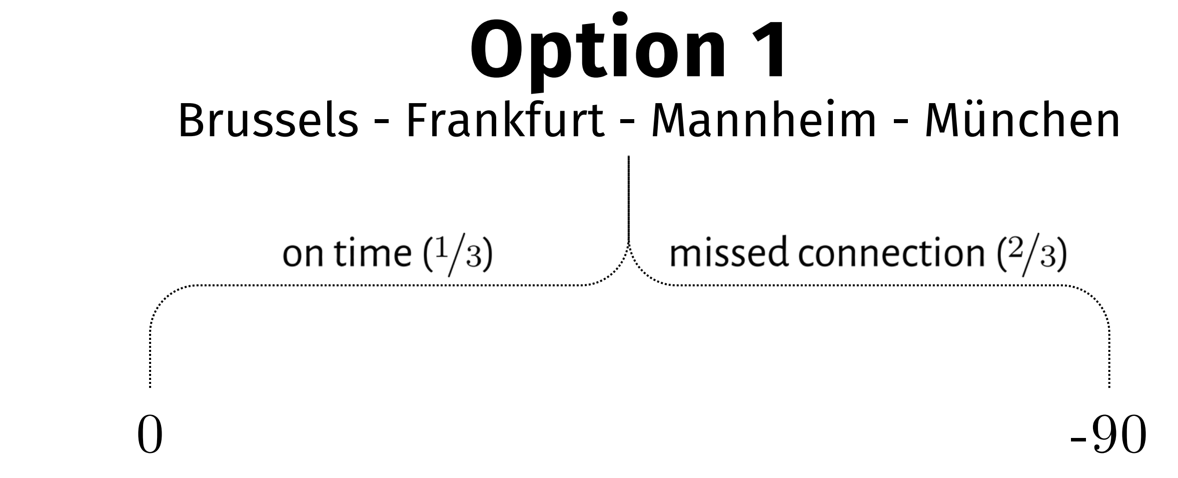
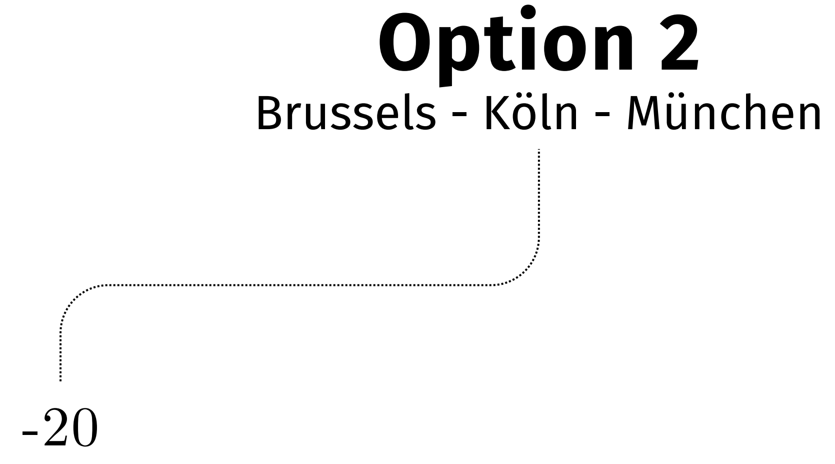


$$\begin{aligned}\mathbb{E}[\text{Route 1}] &= \Pr[\text{on time}] \cdot 0 + \Pr[\text{missed connection}] \cdot (-90) \\ &= {}^1\!/_3 \cdot 0 + {}^2\!/_3 \cdot (-90) \\ &= -60.\end{aligned}$$



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My utility is determined by the arrival time.

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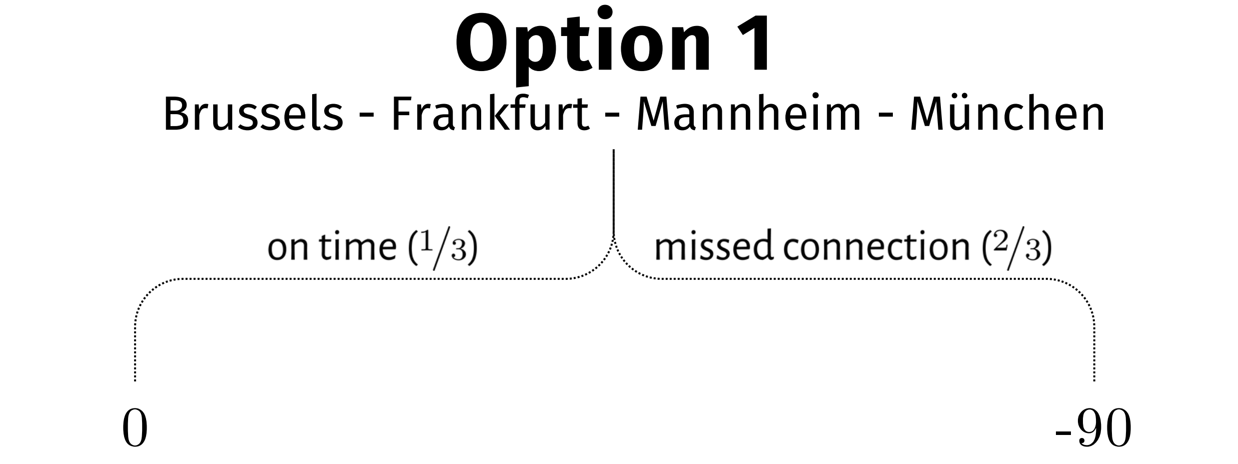
Now the second option seems better.



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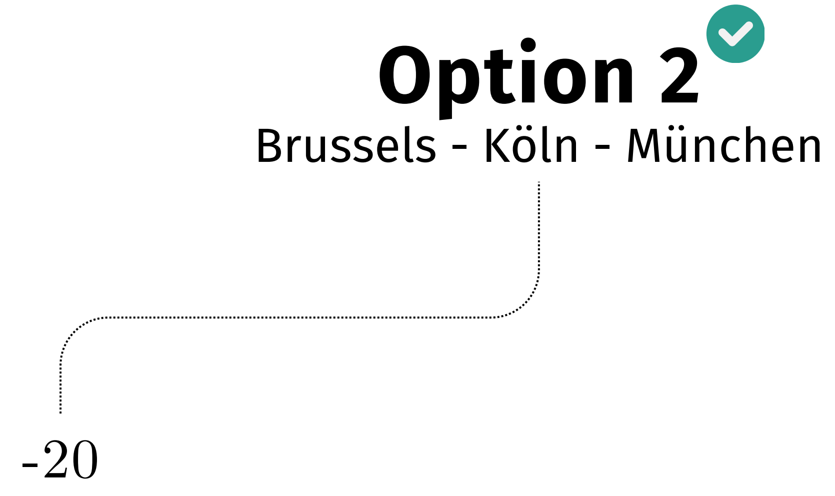

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Now the second option seems better.

But wait! The second option has some uncertainty too: past experience suggests a likely delay.

Option 1

Brussels - Frankfurt - Mannheim - München

on time ($1/3$) missed connection ($2/3$)

0 -90

$$\begin{aligned}\mathbb{E}[\text{Route 1}] &= \Pr[\text{on time}] \cdot 0 + \Pr[\text{missed connection}] \cdot (-90) \\ &= 1/3 \cdot 0 + 2/3 \cdot (-90) \\ &= -60.\end{aligned}$$

Option 2

Brussels - Köln - München

on time ($1/5$) delayed ($4/5$)

-20 -80

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on time ($1/5$) delayed ($4/5$)

-20 -80

$$\begin{aligned}\mathbb{E}[\text{Route 2}] &= \Pr[\text{on time}] \cdot (-20) + \Pr[\text{delayed}] \cdot (-80) \\ &= 1/5 \cdot (-20) + 4/5 \cdot (-80) \\ &= -68.\end{aligned}$$

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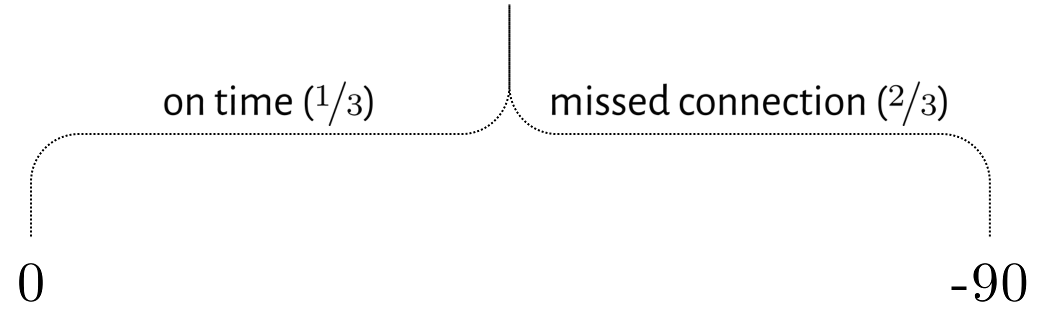
Better to stick with the first option after all...

Option 1

Brussels - Frankfurt - Mannheim - München



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\begin{aligned}
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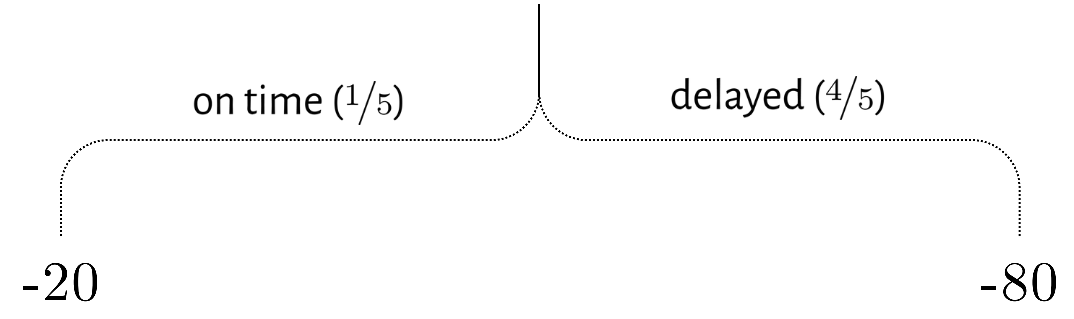
&= -60.
\end{aligned}
$$

Option 2

Brussels - Köln - München



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state of nature

	on time	missed connection in Frankfurt	delay on Köln - München line	...	expected utility*
Option 1	0	-90	0	...	-60
Option 2	-20	0	-80	...	-68

action

utility for taking this action in this state

*Table isn't 100% correct: in general, states need to be mutually exclusive

In general, rational agents (aim to) maximize expected utility.

$$\mathbb{E}\left[u(\text{action})\right] = \sum_{\text{state}} \left(u(\text{action}, \text{state}) \cdot \Pr\left[\text{state}\right] \right)$$

SAM BANKMAN-FRIED

We should be maximising expected value in everything.

And I mean *everything*.

[...] no matter what Bankman-Fried was doing, he was constantly assessing the odds, costs, and benefits.

Faux, Z. (2023). *Number Go Up: Inside Crypto's Wild Rise and Staggering Fall*. Crown Currency.

[...] no matter what Bankman-Fried was doing, he was constantly assessing the odds, costs, and benefits.

Any decision could be boiled down to an “expected value,” [. . .] whether that was a move in a board-game marathon, a billion-dollar trade, or whether to chat with Bezos at a party.

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Bankman-Fried’s goal was always to make as much money as possible, so that he could give it to charity.

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By this metric, even sleep was an unjustifiable luxury. The expected value of staying awake to trade was too high.

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SAM BANKMAN-FRIED

Every minute you spend sleeping is costing you $x dollars, which means you can save fewer lives.

This is not investment advice. Use with caution.*

*Also keep in mind that SBF is in jail today for fraud.

Back to Matching Pennies. Let’s try out some strategies.

TRYING OUT SOME STRATEGIES

Suppose Player 1 uses strategy $s_1 = (0.9, 0.1)$. What should Player 2 do?

payoffs

	Heads	Tails
Heads (0.9)	1, -1	-1, 1
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Pareto optimal strategies

all

pure Nash equilibria

none

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Suppose Player 1 uses strategy $s_1 = (0.9, 0.1)$. What should Player 2 do?

$$\begin{aligned}\mathbb{E}\Big[\text{Heads} \mid s_1\Big] &= (-1)\cdot 0.9 + 1 \cdot 0.1 \\ &= -0.8.\end{aligned}$$

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If Player 2 always plays Heads, i.e., $s_2 = (1, 0)$, they get an average payoff of -0.8. If they always play Tails, i.e., $s_2' = (0, 1)$, they get an average payoff of 0.8.

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Would it make sense for Player 2 to mix between Heads and Tails, say with strategy $s_2'' = (0.3, 0.7)$?

payoffs

	Heads	Tails
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$$\begin{aligned}\mathbb{E}\left[s_2'' \mid s_1\right] &= \mathbb{E}\left[\text{Heads}\right] \cdot 0.3 + \mathbb{E}\left[\text{Tails}\right] \cdot 0.7 \\ &= 0.32 \\ &< 0.8.\end{aligned}$$

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Given that Player 1 plays $s_1 = (0.9, 0.1)$, then, between s_2, s_2' and s_2'', Player 2 would rather play $s_2' = (0, 1)$.

payoffs

	Heads (0)	Tails (1)
Heads (0.9)	1, -1	-1, 1
Tails (0.1)	-1, 1	1, -1

Pareto optimal strategies

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pure Nash equilibria

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Have we found an equilibrium?

AN EQUILIBRIUM?

If Player 1 plays $s_1 = (0.9, 0.1)$, Player 2's maximizes expected utility by playing $s'_2 = (0, 1)$ (easy to check!).

payoffs

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AN EQUILIBRIUM?

If Player 1 plays $s_1 = (0.9, 0.1)$, Player 2's maximizes expected utility by playing $s_2' = (0, 1)$ (easy to check!).

In other words, $s_2' = (0, 1)$ is Player 2's *best response* to $s_1 = (0.9, 0.1)$.

payoffs

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So is $\boldsymbol{s} = (s_1, s_2')$ a Nash equilibrium?

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So is $\boldsymbol{s} = (s_1, s_2')$ a Nash equilibrium?

No! If Player 2 plays $s_2' = (0, 1)$, Player 1 wants to deviate to $s_1' = (0, 1)$:

$$\begin{aligned}\mathbb{E}\Big[s_1 \mid s_2'\Big] &= \mathbb{E}\Big[\text{Heads} \mid s_2'\Big] \cdot 0.9 + \mathbb{E}\Big[\text{Tails} \mid s_2'\Big] \cdot 0.1 \\ &= (-1) \cdot 0.9 + 1 \cdot 0.1 \\ &= -0.8\end{aligned}$$

payoffs

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In other words, $s_2' = (0, 1)$ is Player 2's *best response* to $s_1 = (0.9, 0.1)$.

So is $s = (s_1, s_2')$ a Nash equilibrium?

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&= (-1) \cdot 0 + 1 \cdot 1 \\
&= 1 \\
&> \mathbb{E}\left[s_1 \mid s_2'\right].
\end{aligned}
$$

payoffs

	Heads (0)	Tails (1)
Heads (0.9)	1, -1	-1, 1
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Pareto optimal strategies

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pure Nash equilibria

none

Let's find a mixed equilibrium.

FINDING MIXED EQUILIBRIA

Suppose Players 1 and 2 play mixed strategies $s_1 = (p, 1-p)$ and $s_2 = (q, 1-q)$, respectively, for $p, q > 0$.

payoffs

	Heads (q)	Tails (1 - q)
Heads (p)	1, -1	-1, 1
Tails (1 - p)	-1, 1	1, -1

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mixed Nash equilibria

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Note that Player 2's expected payoff with these strategies is:

$$\mathbb{E}\Big[s_2 \mid s_1\Big] = \mathbb{E}\Big[\text{Heads} \mid s_1\Big] \cdot q + \mathbb{E}\Big[\text{Tails} \mid s_1\Big] \cdot (1-q).$$

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Suppose, now, that Player 1's strategy makes Heads more attractive for Player 2:

$$\mathbb{E}\Big[\text{Heads} \mid s_1\Big] > \mathbb{E}\Big[\text{Tails} \mid s_1\Big].$$

payoffs

	Heads (q)	Tails (1 - q)
Heads (p)	1, -1	-1, 1
Tails (1 - p)	-1, 1	1, -1

Pareto optimal strategies

all

pure Nash equilibria

none

mixed Nash equilibria

FINDING MIXED EQUILIBRIA

Suppose Players 1 and 2 play mixed strategies $s_1 = (p, 1-p)$ and $s_2 = (q, 1-q)$, respectively, for $p, q > 0$.

Note that Player 2's expected payoff with these strategies is:

$$\mathbb{E}\Big[s_2 \mid s_1\Big] = \mathbb{E}\Big[\text{Heads} \mid s_1\Big] \cdot q + \mathbb{E}\Big[\text{Tails} \mid s_1\Big] \cdot (1-q).$$

Suppose, now, that Player 1's strategy makes Heads more attractive for Player 2:

$$\mathbb{E}\Big[\text{Heads} \mid s_1\Big] > \mathbb{E}\Big[\text{Tails} \mid s_1\Big].$$

In this case, Player 2 would want to deviate to $s_2' = (1, 0)$:

$$\begin{aligned} \mathbb{E}\Big[s_2' \mid s_1\Big] &= \mathbb{E}\Big[\text{Heads} \mid s_1\Big] \cdot 1 + \mathbb{E}\Big[\text{Tails} \mid s_1\Big] \cdot 0 \\ &> \mathbb{E}\Big[\text{Heads} \mid s_1\Big] \cdot q + \mathbb{E}\Big[\text{Tails} \mid s_1\Big] \cdot (1-q). \end{aligned}$$

payoffs

	Heads (q)	Tails (1 - q)
Heads (p)	1, -1	-1, 1
Tails (1 - p)	-1, 1	1, -1

Pareto optimal strategies

all

pure Nash equilibria

none

mixed Nash equilibria

FINDING MIXED EQUILIBRIA

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So $s = (s_1, s_2)$ cannot be a Nash equilibrium. Same if $\mathbb{E}\Big[\text{Tails} \mid s_1\Big] > \mathbb{E}\Big[\text{Heads} \mid s_1\Big]$.

payoffs

	Heads (q)	Tails (1 - q)
Heads (p)	1, -1	-1, 1
Tails (1 - p)	-1, 1	1, -1

Pareto optimal strategies

all

pure Nash equilibria

none

mixed Nash equilibria

2/2

FINDING MIXED EQUILIBRIA

Suppose Players 1 and 2 play mixed strategies $s_1 = (p, 1-p)$ and $s_2 = (q, 1-q)$, respectively, for $p, q > 0$.

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The only way to avoid this is for Player 1 to play a strategy $s_1^* = (p, 1-p)$ that makes Player 2 indifferent between their actions:

$$\begin{aligned} \mathbb{E}\big[\text{Heads} \mid s_1\big] = \mathbb{E}\big[\text{Tails} \mid s_1\big] \ &\text{iff } (-1) \cdot p + 1 \cdot (1-p) = 1 \cdot p + (-1) \cdot (1-p) \\ &\text{iff } p = 1/2. \end{aligned}$$

payoffs

	Heads (q)	Tails (1 - q)
Heads (p)	1, -1	-1, 1
Tails (1 - p)	-1, 1	1, -1

Pareto optimal strategies

all

pure Nash equilibria

none

mixed Nash equilibria

FINDING MIXED EQUILIBRIA

Suppose Players 1 and 2 play mixed strategies $s_1 = (p, 1-p)$ and $s_2 = (q, 1-q)$, respectively, for $p, q > 0$.

Note that Player 2's expected payoff with these strategies is:

$$\mathbb{E}\Big[s_2 \mid s_1\Big] = \mathbb{E}\Big[\text{Heads} \mid s_1\Big] \cdot q + \mathbb{E}\Big[\text{Tails} \mid s_1\Big] \cdot (1-q).$$

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So Player 1 wants to play $s_1^* = (1/2, 1/2)$. Similarly, Player 2 wants to play $s_2^* = (1/2, 1/2)$. This is the mixed Nash equilibrium.

payoffs

	Heads (q)	Tails (1 - q)
Heads (p)	1, -1	-1, 1
Tails (1 - p)	-1, 1	1, -1

Pareto optimal strategies

all

pure Nash equilibria

none

mixed Nash equilibria

$$s^* = \Big((1/2, 1/2), (1/2, 1/2)\Big)$$

Key takeaway: in a mixed equilibrium, you’re indifferent between your actions.

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In Matching Pennies everyone gets, on average, 0.

Key takeaway: in a mixed equilibrium, you're indifferent between your actions.

In Matching Pennies everyone gets, on average, 0. But deviating from this would get you less.

JOHN NASH

This can be generalized to any game with finitely many actions.

NASH'S THEOREM

THEOREM (NASH, 1951)

Any game with a finite number of players and finite actions has a Nash equilibrium in mixed strategies.

Nash, J. (1951). Non-Cooperative Games. *Annals of Mathematics*, 54(2), 286–295.

JOHN NASH

They gave me the Nobel prize for this result!

The moral is that sometimes pure equilibria are useless. You need to make yourself unpredictable.

Fun fact: humans are not that good at randomizing.

ARIEL RUBINSTEIN

In experiments, they keep trying to detect patterns, are susceptible to stories and framing effects.

Mookherjee, D., & Sopher, B. (1994). Learning Behavior in an Experimental Matching Pennies Game. *Games and Economic Behavior*, 7(1), 62–91.

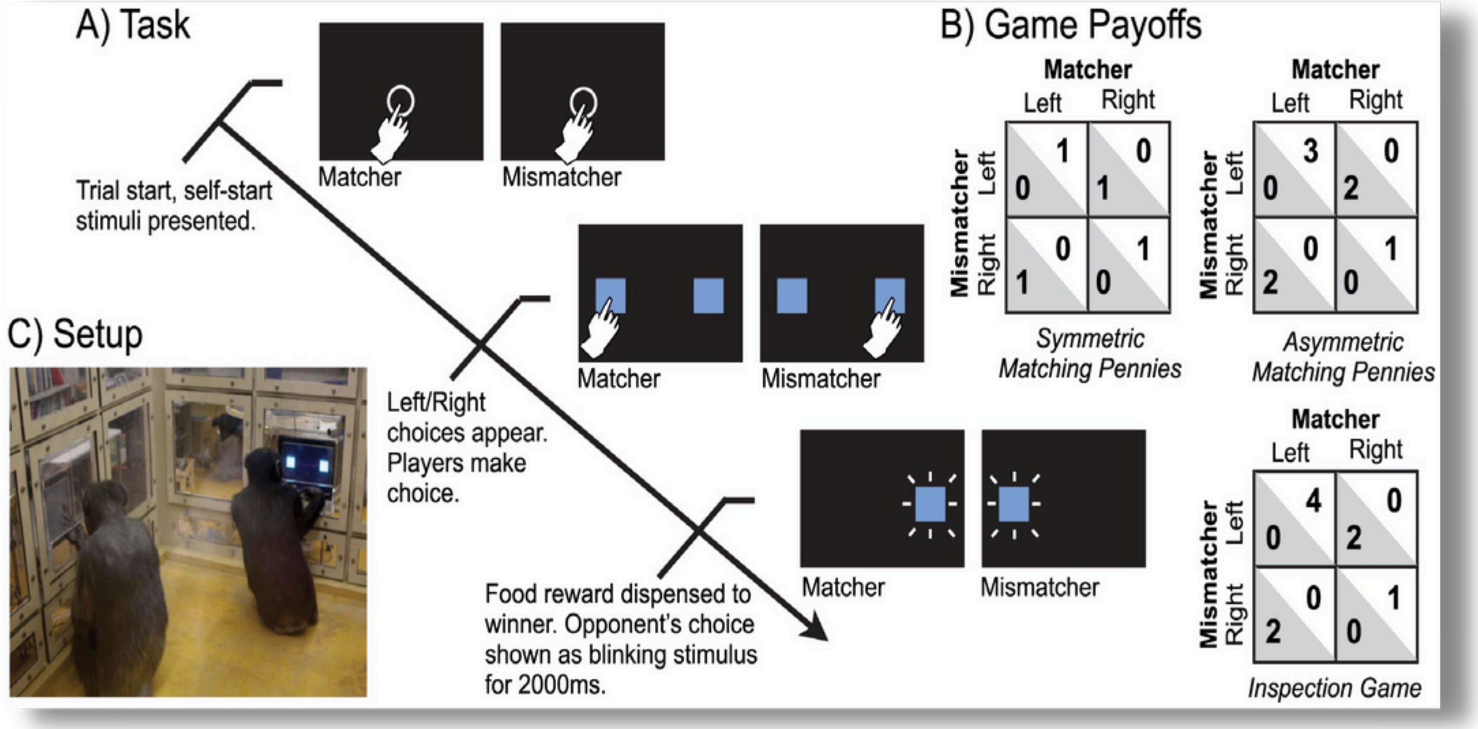
Eliaz, K., & Rubinstein, A. (2011). Edgar Allan Poe's riddle: Framing effects in repeated matching pennies games. *Games and Economic Behavior*, 71(1), 88–99.

But chimpanzees seem pretty good at it.

MATCHING PENNIES WITH CHIMPANZEES

COLIN CAMERER

In a matching pennies experiment, chimpanzees were quite good at approximating the Nash equilibrium.



Martin, C. F., Bhui, R., Bossaerts, P., Matsuzawa, T., & Camerer, C. (2014). Chimpanzee choice rates in competitive games match equilibrium game theory predictions. *Nature: Scientific Reports*, 4, 5182.