LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

JUNE 30, 2025

NETWORKED MINDS:
OPINION DYNAMICS AND COLLECTIVE INTELLIGENCE IN SOCIAL NETWORKS

NAIVE LEARNING & WISE CROWDS

Adrian Haret
a.haret@lmu.de

Quiz time!

What is the population density (nr people/km^2) of Munich, approximately?

- ☐ 21,000
- ☐ 7,500
- ☐ 4,900
- ☐ 4,100
- ☐ 3,000
- ☐ 2,100

Feel free to discuss!

What is the population density (nr people/km^2) of Munich, approximately?

- ☐ 21,000
- ☐ 7,500
- ☐ 4,900
- ☐ 4,100
- ☐ 3,000
- ☐ 2,100

What is the population density (nr people/km^2) of Munich, approximately?

- ☐ 21,000 PARIS
- ☐ 7,500
- ☐ 4,900
- ☐ 4,100
- ☐ 3,000
- ☐ 2,100

Wikipedia contributors. (2025, March 23). List of European Union cities proper by population density. Wikipedia.

What is the population density (nr people/km^2) of Munich, approximately?

- ☐ 21,000 PARIS
- ☐ 7,500 MILAN
- ☐ 4,900
- ☐ 4,100
- ☐ 3,000
- ☐ 2,100

Wikipedia contributors. (2025, March 23). List of European Union cities proper by population density. Wikipedia.

What is the population density (nr people/km^2) of Munich, approximately?

- ☐ 21,000 PARIS
- ☐ 7,500 MILAN
- ☑ 4,900 MUNICH
- ☐ 4,100
- ☐ 3,000
- ☐ 2,100

Wikipedia contributors. (2025, March 23). List of European Union cities proper by population density. Wikipedia.

What is the population density (nr people/km^2) of Munich, approximately?

- [ ] 21,000 PARIS
- [ ] 7,500 MILAN
- [x] 4,900 MUNICH
- [ ] 4,100 BERLIN
- [ ] 3,000
- [ ] 2,100

Wikipedia contributors. (2025, March 23). List of European Union cities proper by population density. Wikipedia.

What is the population density (nr people/km^2) of Munich, approximately?

- ☐ 21,000 PARIS
- ☐ 7,500 MILAN
- ☑ 4,900 MUNICH
- ☐ 4,100 BERLIN
- ☐ 3,000 FRANKFURT
- ☐ 2,100

Wikipedia contributors. (2025, March 23). List of European Union cities proper by population density. Wikipedia.

What is the population density (nr people/km^2) of Munich, approximately?

- ☐ 21,000 PARIS
- ☐ 7,500 MILAN
- ☑ 4,900 MUNICH
- ☐ 4,100 BERLIN
- ☐ 3,000 FRANKFURT
- ☐ 2,100 ROME

Wikipedia contributors. (2025, March 23). List of European Union cities proper by population density. Wikipedia.

Can social influence make things go awry with collective beliefs?

AD 1330

ODORIC OF PORDENONE

In a province of the Grand Can there grow gourds, which, when they are ripe, open, and within them is found a little beast like unto a young lamb...

Odoric of Pordenone [*trans.* Sir Henry Yule] (2002). *The Travels of Friar Odoric.* W.B. Eerdmans Publishing Company.

AD 1357 - 1371

SIR JOHN MANDEVILLE

In Tartary groweth a manner of fruit, as though it were gourds. And when they be ripe, men cut them a-two, and find within a little beast, in flesh, in bone, and blood, as though it were a little lamb without wool. And men eat both the fruit and the beast. And that is a great marvel.

Mandeville, J. (1900). *The Travels of Sir John Mandeville. The Cotton Manuscript in modern spelling.* Macmillan and Co. Limited.

AD 1357 - 1371

SIR JOHN MANDEVILLE

In Tartary groweth a manner of fruit, as though it were gourds. And when they be ripe, men cut them a-two, and find within a little beast, in flesh, in bone, and blood, as though it were a little lamb without wool. And men eat both the fruit and the beast. And that is a great marvel.

Of that fruit I have eaten… and found it wondirfulle.

Mandeville, J. (1900). *The Travels of Sir John Mandeville. The Cotton Manuscript in modern spelling.* Macmillan and Co. Limited.

AD 1515 - 1553

BARON SIGISMUND VON HERBERSTEIN

[...] a certain seed like that of a melon, but rather rounder and longer, from which, when it was set in the earth, grew a plant resembling a lamb, and attaining to a height of about two and a half feet...

Sigmund Freiherr von Herberstein (1851). Notes Upon Russia: Being a Translation of the Earliest Account of that Country, Entitled Rerum Moscoviticarum Commentarii. Hakluyt Society.

AD 1605


STOIRE
DMIRABLE DES
PLANTES ET HERBES ESMER-
ueillables & miraculeuses en nature : mesmes
d'aucunes qui sont vrays Zoophytes, ou Plant'-
animales, Plantes & Animaux tout ensemble,
pour auoir vie vegetatiue, sensitiue & animale:

Auec leurs Portraicts au naturel, selon les histoires, descriptions, voyages, & nauigations des anciens & modernes Hebrieux, Chaldees, Egyptiens, Assyriens, Armeniens, Grecs, Latins, Africains, Arabes, Nubiens, Ethyopiens, Sarrasins, Turcs, Mores, Persans, Tartares, Chinois, Indiens, Portugays, Espagnols, François, Flaments, Anglois, Polonois, Moschouites, Allemans, & autres.

Par M. CLAVDE DVRET, President à
Moulins en Bourb...

CLAUDE DURET

Duret, C. (1605). Histoire Admirable des Plantes.

AD 1641

ATHANASIUS KIRCHER

[...] we assert that it is a plant. Though its form be that of a quadruped, and the juice beneath its woolly covering be blood which flows if an incision be made in its flesh, these things will not move us. It will be found to be a plant.

Kircher, A. (1641). *Magnes; sive de arte magneticâ opus tripartitum.*

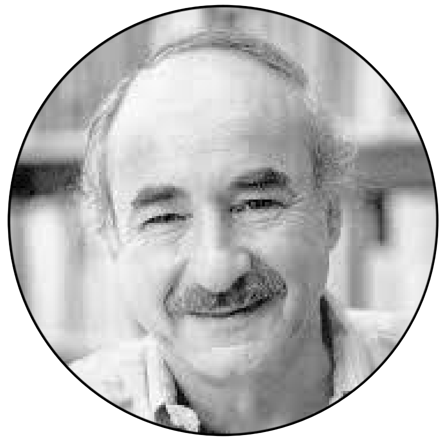
AD 1683

ENGELBERT KAEMPFER

I *have searched ad risum et nauseam for this zoophyte feeding on grass, but have found nothing.*

Kaempfer, E. (1712). *Amœnitatum* Exoticarum *politico-physico-medicarum fascicul.*

Let's model this.



MORRIS DEGROOT

Agents are represented by nodes in a social network.

And they update their opinions depending on the opinions of their peers.

THE DEGROOT MODEL

There is a set $N = \{1, 2, \ldots, n\}$ of *agents*.

THE DEGROOT MODEL

There is a set $N = \{1, 2, \ldots, n\}$ of *agents*. Each agent i has an *opinion*, or *belief*, $x_i \in [0, 1]$.

THE DEGROOT MODEL

There is a set $N = \{1, 2, \ldots, n\}$ of *agents*. Each agent i has an *opinion*, or *belief*, $x_i \in [0, 1]$. The opinions are meant to track a *true state* $\mu \in (0, 1)$.

THE DEGROOT MODEL

There is a set $N = \{1, 2, \ldots, n\}$ of *agents*. Each agent i has an *opinion*, or *belief*, $x_i \in [0, 1]$. The opinions are meant to track a *true state* $\mu \in (0, 1)$.

Time goes by in discrete steps $t \in \{0, 1, 2, \ldots\}$. Agent i's *opinion at time* t is x_i^t.

THE DEGROOT MODEL

There is a set $N = \{1, 2, \ldots, n\}$ of *agents*. Each agent i has an *opinion*, or *belief*, $x_i \in [0, 1]$. The opinions are meant to track a *true state* $\mu \in (0, 1)$.

Time goes by in discrete steps $t \in \{0, 1, 2, \ldots\}$. Agent i's *opinion at time* t is x_i^t.

Agents are connected by a *social network* $G = (N, E)$, which is a directed graph. An edge from i to j indicates that agent i pays attention to agent j.

THE DEGROOT MODEL

There is a set $N = \{1, 2, \ldots, n\}$ of *agents*. Each agent i has an *opinion*, or *belief*, $x_i \in [0, 1]$. The opinions are meant to track a *true state* $\mu \in (0, 1)$.

Time goes by in discrete steps $t \in \{0, 1, 2, \ldots\}$. Agent i's *opinion at time* t is x_i^t.

Agents are connected by a *social network* $G = (N, E)$, which is a directed graph. An edge from i to j indicates that agent i pays attention to agent j. Agent i's *(out-)neighborhood* $N(i)$ is the set of agents that i pays attention to:

$$N(i) = \{j \in N \mid (i, j) \in E\}.$$

THE DEGROOT MODEL

There is a set $N = \{1, 2, \ldots, n\}$ of *agents*. Each agent i has an *opinion*, or *belief*, $x_i \in [0, 1]$. The opinions are meant to track a *true state* $\mu \in (0, 1)$.

Time goes by in discrete steps $t \in \{0, 1, 2, \ldots\}$. Agent i's *opinion at time* t is x_i^t.

Agents are connected by a *social network* $G = (N, E)$, which is a directed graph. An edge from i to j indicates that agent i pays attention to agent j. Agent i's *(out-)neighborhood* $N(i)$ is the set of agents that i pays attention to:

$$N(i) = \{j \in N \mid (i, j) \in E\}.$$

Each agent i distributes a total weight of 1 across the agents in $N(i)$:

$$\sum_{j \in N(i)} w_{ij} = 1,$$

where $w_{ij} > 0$ is the *weight* that agent i places on agent j's opinion.

THE DEGROOT MODEL

There is a set $N = \{1, 2, \ldots, n\}$ of *agents*. Each agent i has an *opinion*, or *belief*, $x_i \in [0, 1]$. The opinions are meant to track a *true state* $\mu \in (0, 1)$.

Time goes by in discrete steps $t \in \{0, 1, 2, \ldots\}$. Agent i's *opinion at time* t is x_i^t.

Agents are connected by a *social network* $G = (N, E)$, which is a directed graph. An edge from i to j indicates that agent i pays attention to agent j. Agent i's *(out-)neighborhood* $N(i)$ is the set of agents that i pays attention to:

$$N(i) = \{j \in N \mid (i, j) \in E\}.$$

Each agent i distributes a total weight of 1 across the agents in $N(i)$:

$$\sum_{j \in N(i)} w_{ij} = 1,$$

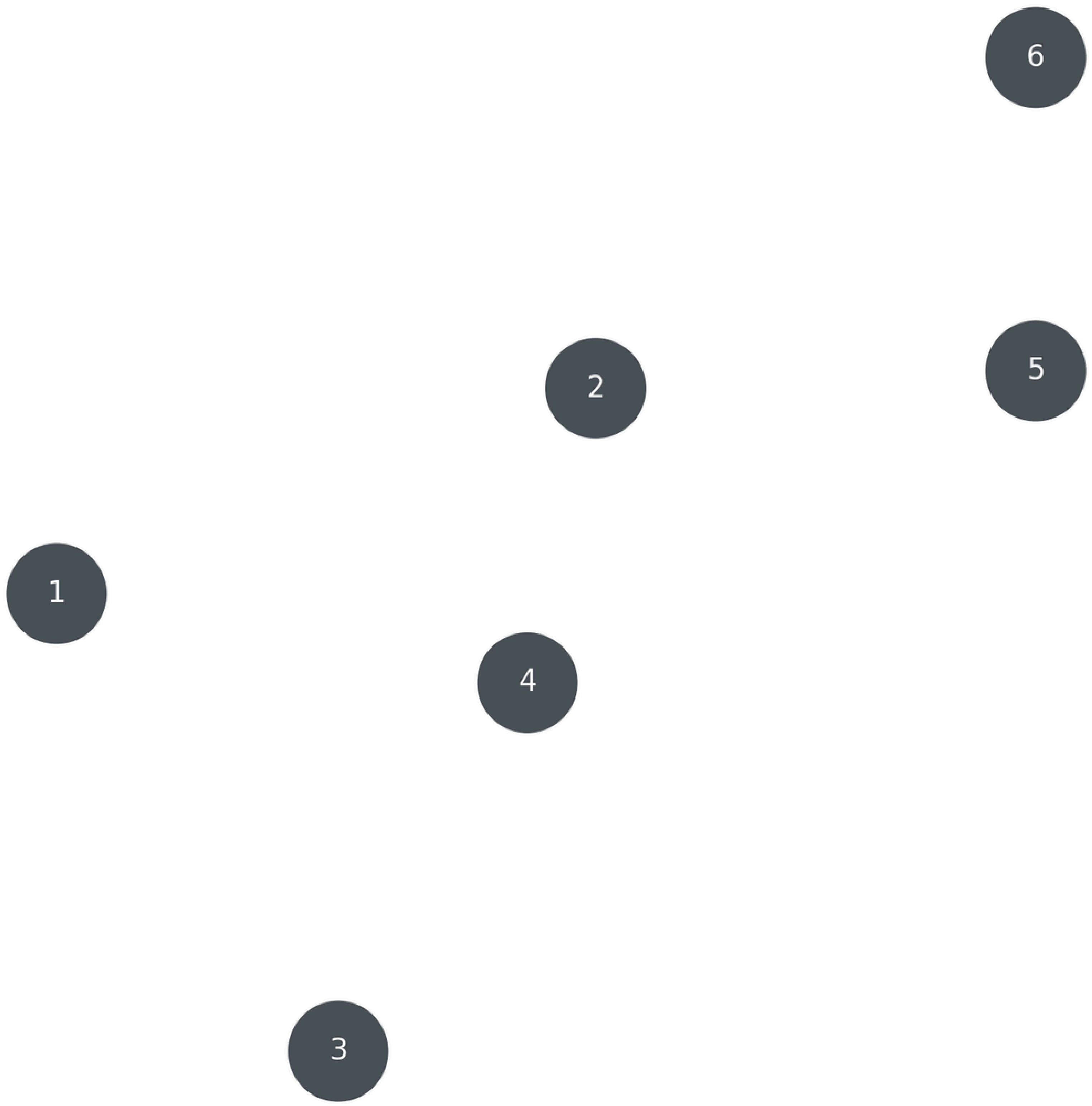
where $w_{ij} > 0$ is the *weight* that agent i places on agent j's opinion.

At each new time step, agents *update* their opinions to a weighted average of the opinions of agents they pay attention to:

$$x_i^{t+1} = \sum_{j \in N(i)} w_{ij} x_j^t.$$

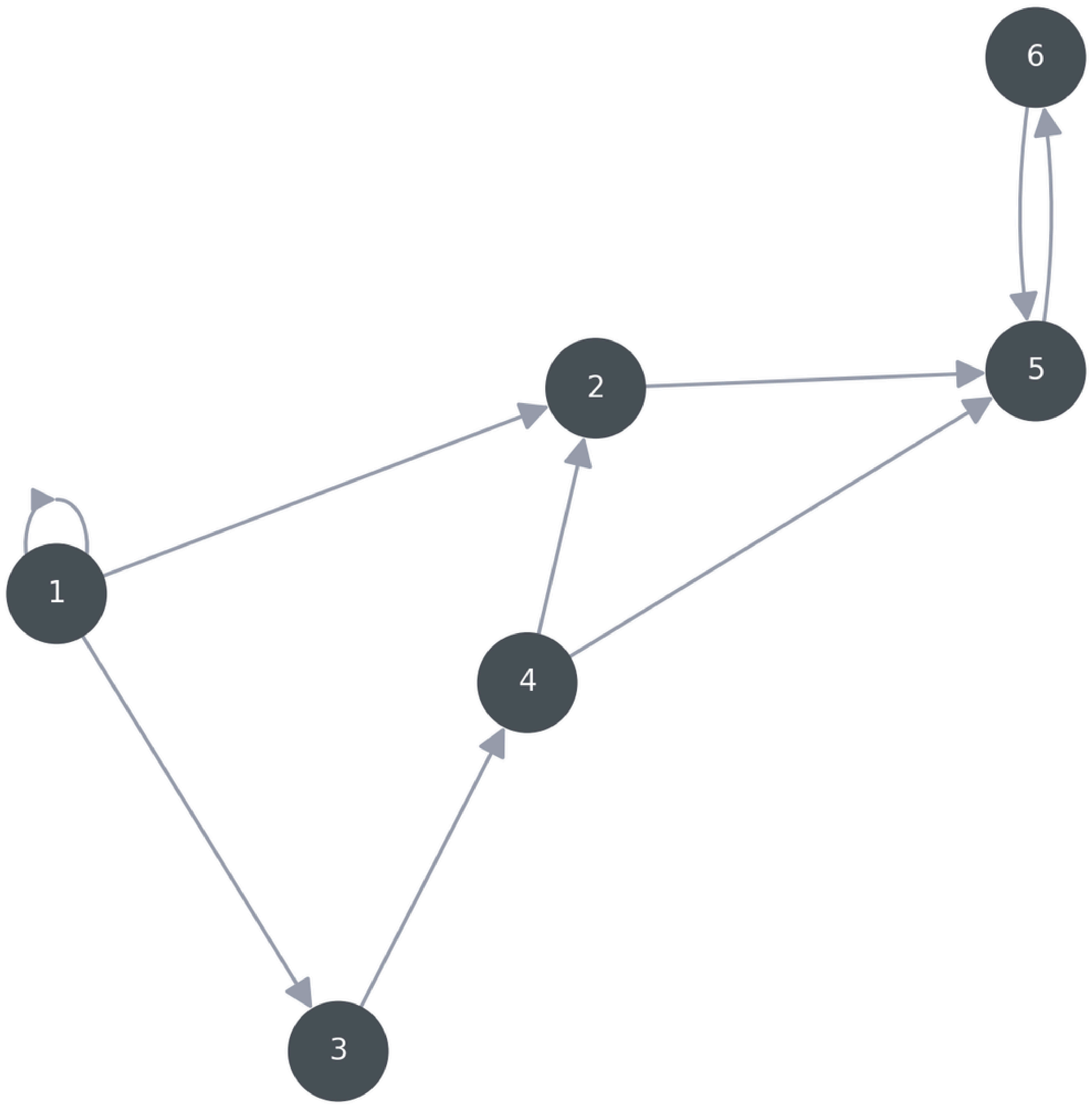
THE DEGROOT MODEL: EXAMPLE

Take $N = \{1, \ldots, 6\}$ to be the set of agents.



THE DEGROOT MODEL: EXAMPLE

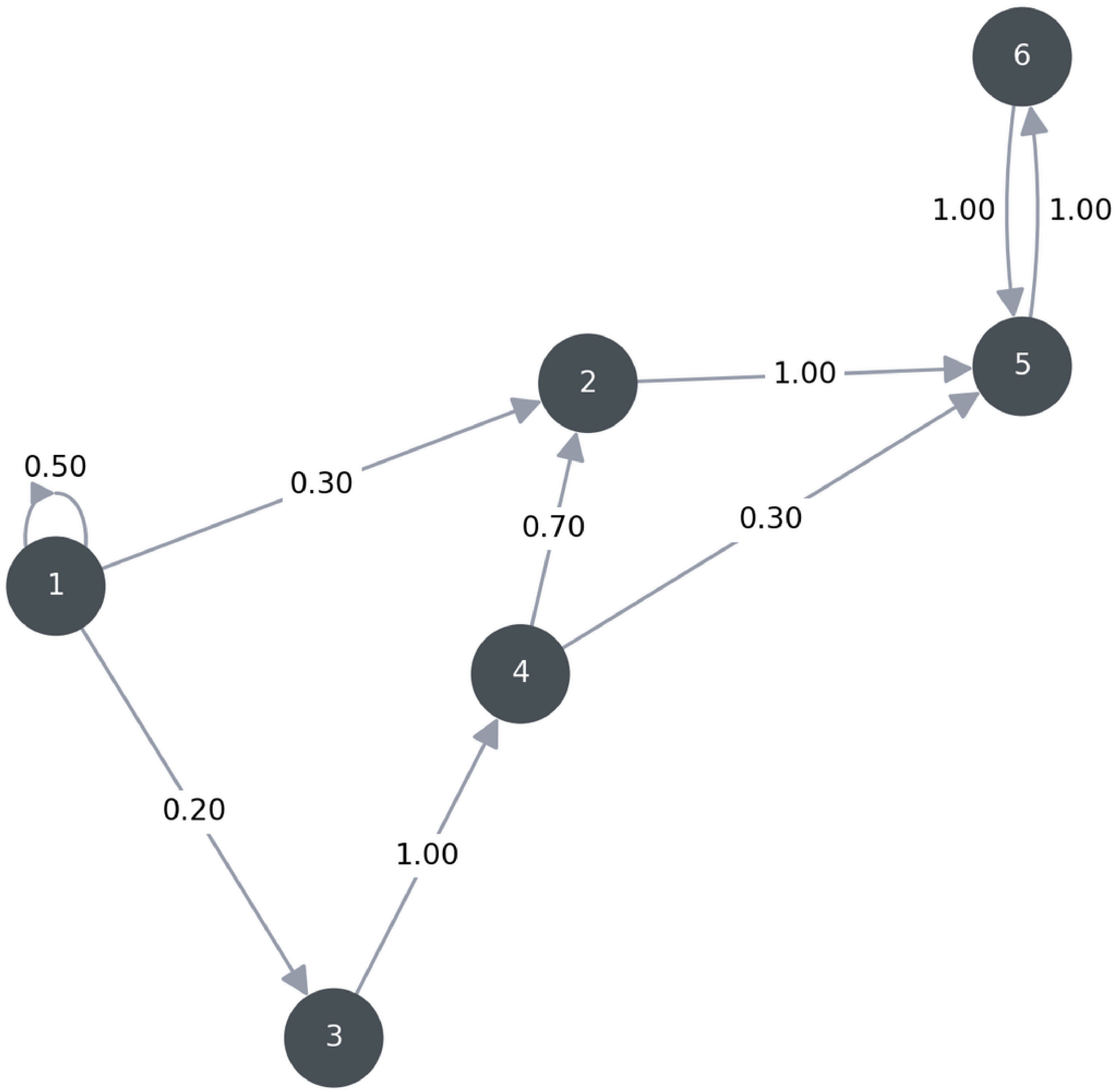
Take $N = \{1, ..., 6\}$ to be the set of agents. They are connected by the social network G on the right.



THE DEGROOT MODEL: EXAMPLE

Take $N = \{1, \ldots, 6\}$ to be the set of agents. They are connected by the social network G on the right.

Each agent distributes a total weight of 1 across the agents in their neighborhood.

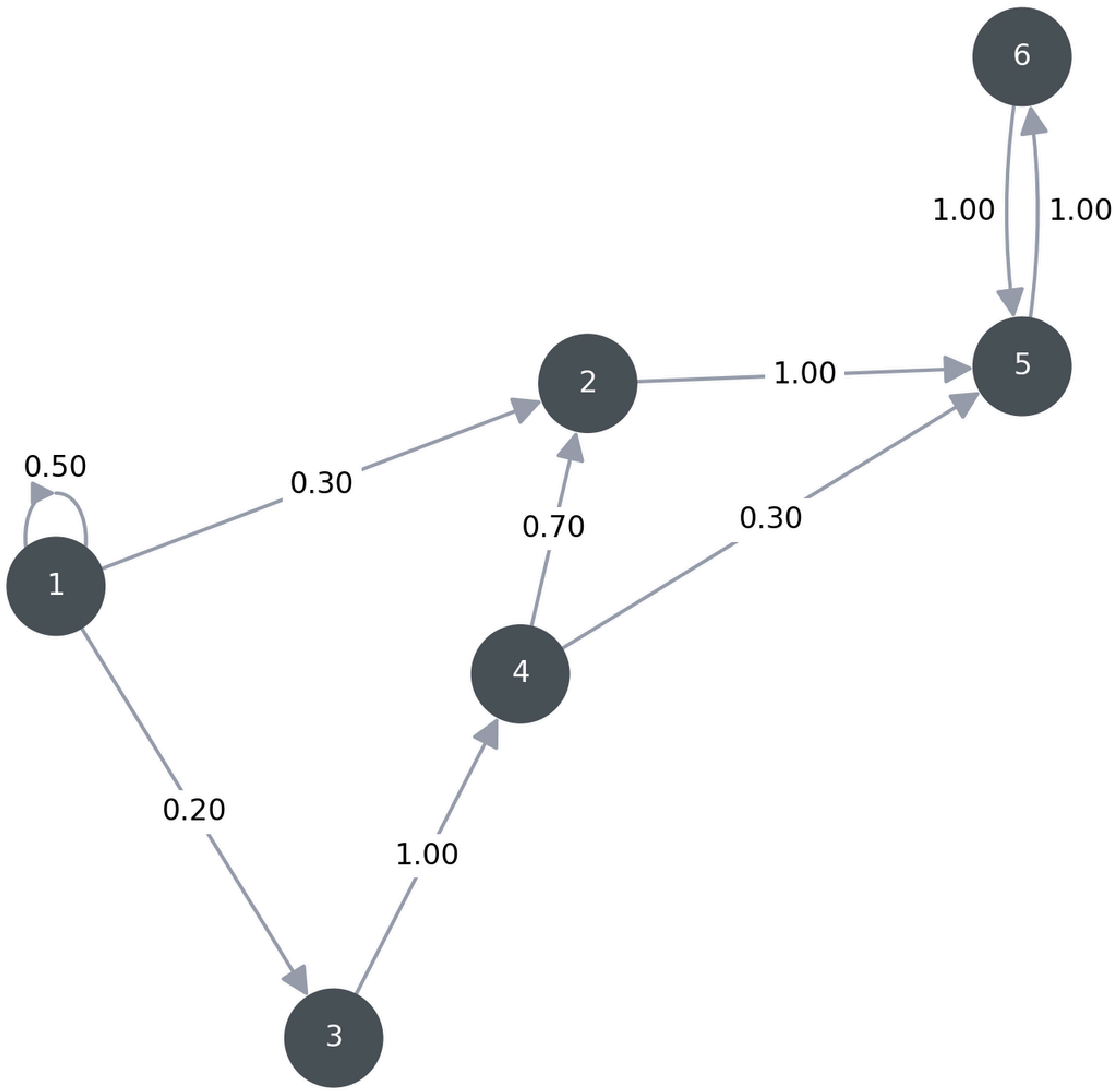


THE DEGROOT MODEL: EXAMPLE

Take $N = \{1, \ldots, 6\}$ to be the set of agents. They are connected by the social network G on the right.

Each agent distributes a total weight of 1 across the agents in their neighborhood.

The true state is $\mu = 0.5$.

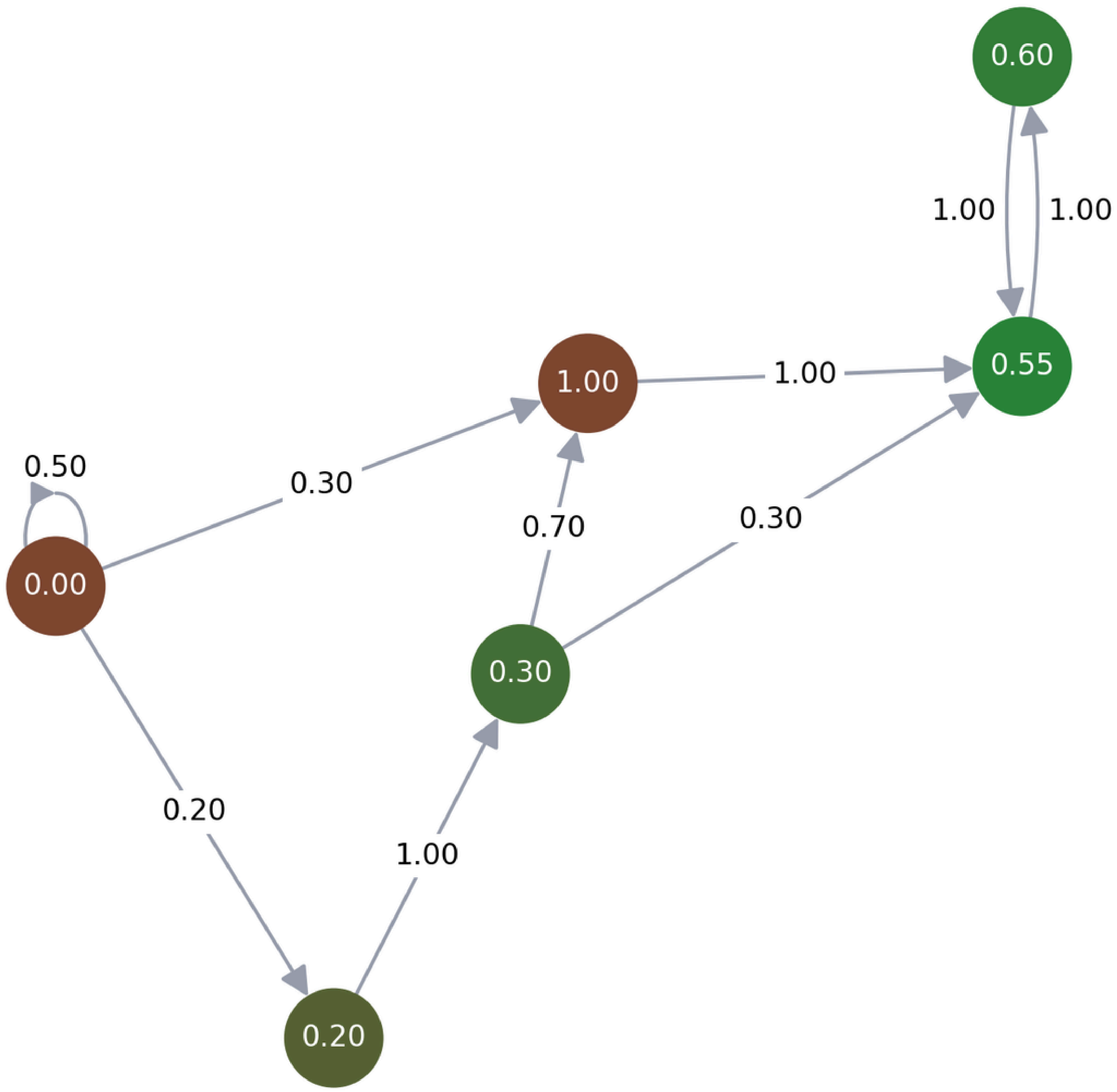


THE DEGROOT MODEL: EXAMPLE

Take $N = \{1, \ldots, 6\}$ to be the set of agents. They are connected by the social network G on the right.

Each agent distributes a total weight of 1 across the agents in their neighborhood.

The true state is $\mu = 0.5$. Each agent starts with the initial belief shown on the right.*



*Greener means closer to the truth, redder is farther away.

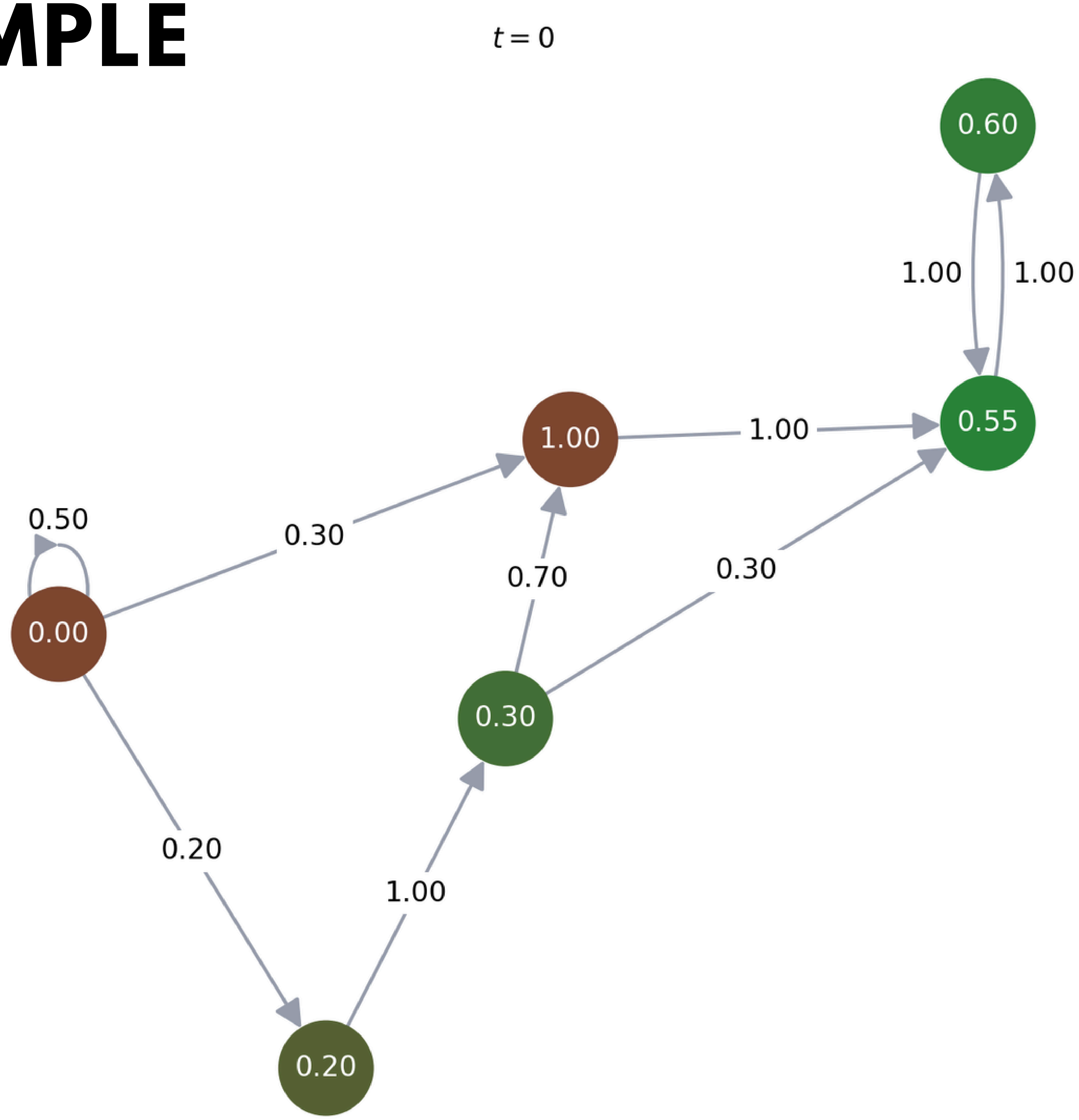
THE DEGROOT MODEL: EXAMPLE

Take $N = \{1, \ldots, 6\}$ to be the set of agents. They are connected by the social network G on the right.

Each agent distributes a total weight of 1 across the agents in their neighborhood.

The true state is $\mu = 0.5$. Each agent starts with the initial belief shown on the right.*

Time starts at $t = 0$.



*Greener means closer to the truth, redder is farther away.

THE DEGROOT MODEL: EXAMPLE

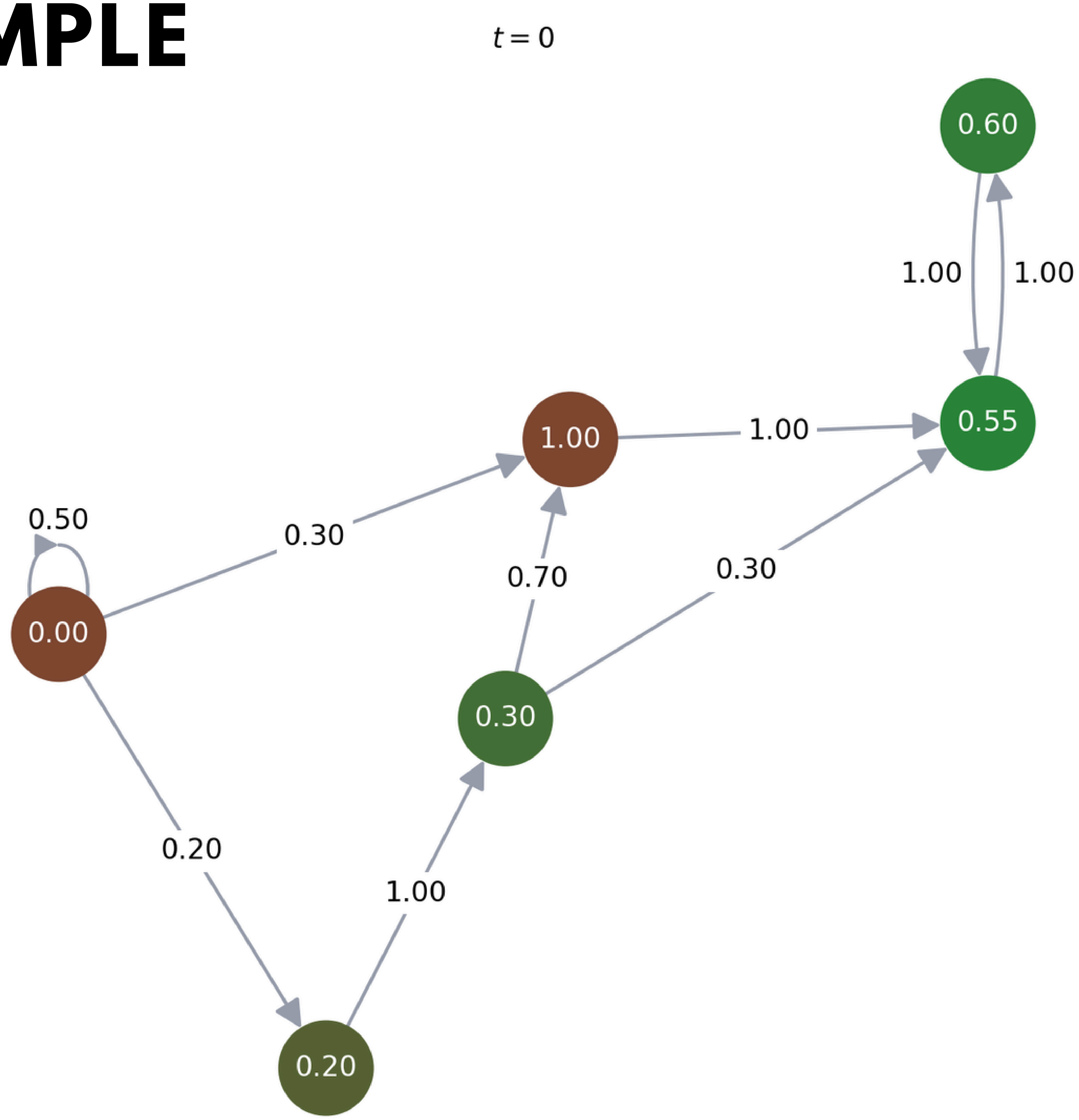
Take $N = \{1, \ldots, 6\}$ to be the set of agents. They are connected by the social network G on the right.

Each agent distributes a total weight of 1 across the agents in their neighborhood.

The true state is $\mu = 0.5$. Each agent starts with the initial belief shown on the right.*

Time starts at $t = 0$.

At $t = 1$, each agent updates their belief to a weighted average of the beliefs of agents they pay attention to.



*Greener means closer to the truth, redder is farther away.

THE DEGROOT MODEL: EXAMPLE

Take $N = \{1, \ldots, 6\}$ to be the set of agents. They are connected by the social network G on the right.

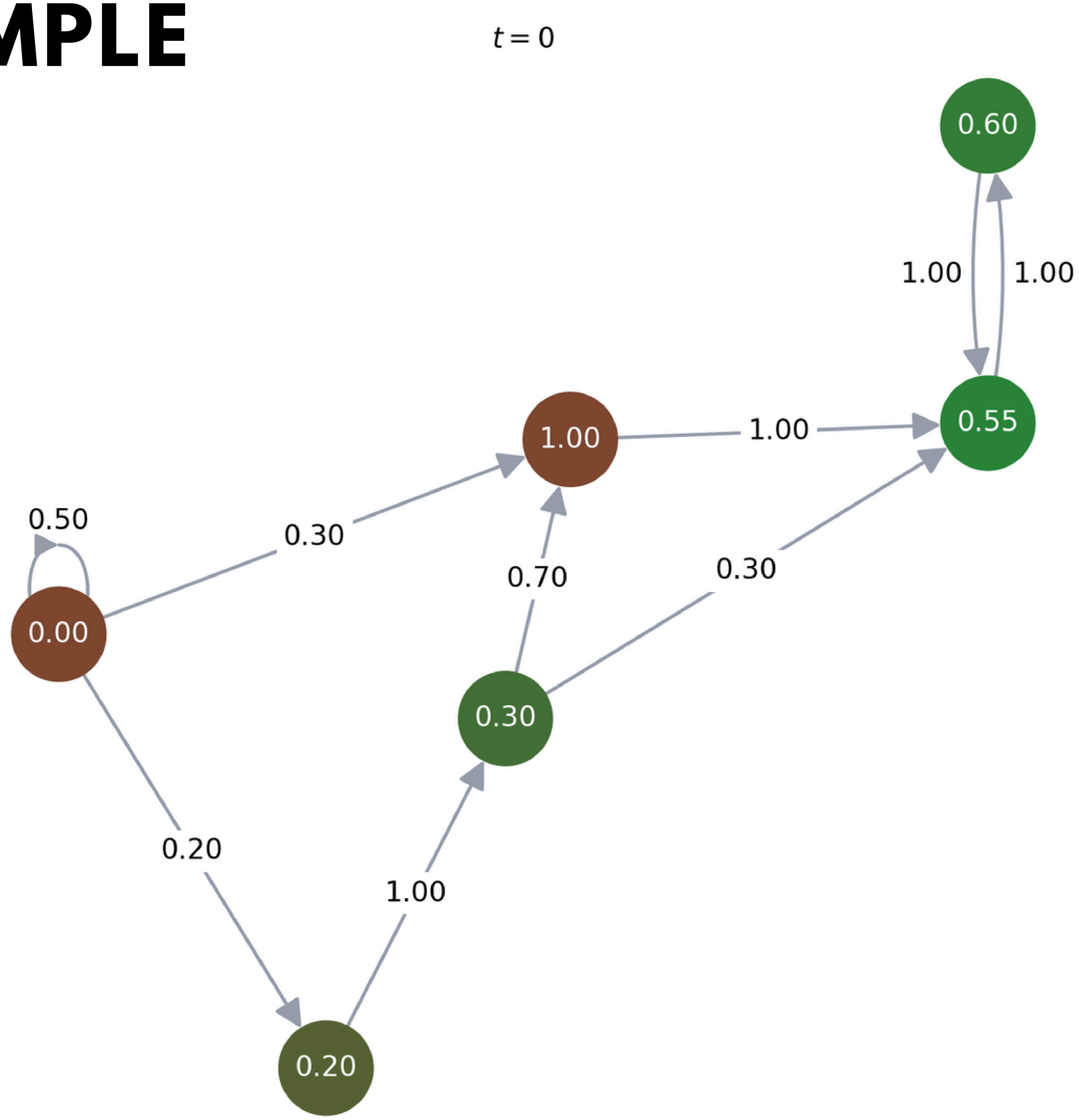
Each agent distributes a total weight of 1 across the agents in their neighborhood.

The true state is $\mu = 0.5$. Each agent starts with the initial belief shown on the right.*

Time starts at $t = 0$.

At $t = 1$, each agent updates their belief to a weighted average of the beliefs of agents they pay attention to. For instance, agent 1's belief becomes:

$$
\begin{aligned}
x_1^1 &= 0.5 \cdot 0 + 0.3 \cdot 1 + 0.2 \cdot 0.2 \\
&= 0.34.
\end{aligned}
$$



*Greener means closer to the truth, redder is farther away.

THE DEGROOT MODEL: EXAMPLE

Take $N = \{1, \dots, 6\}$ to be the set of agents. They are connected by the social network G on the right.

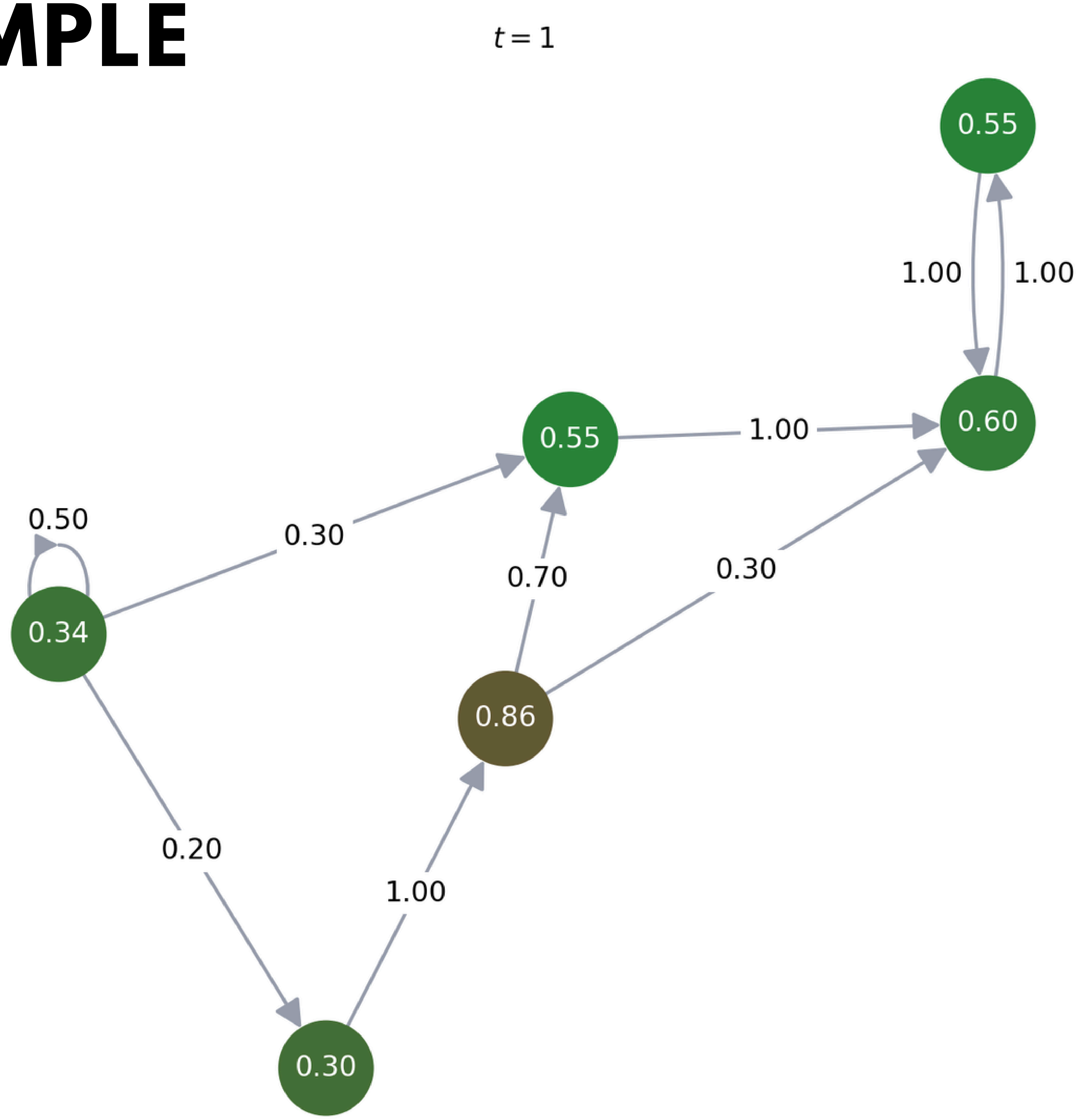
Each agent distributes a total weight of 1 across the agents in their neighborhood.

The true state is $\mu = 0.5$. Each agent starts with the initial belief shown on the right.*

Time starts at $t = 0$.

At $t = 1$, each agent updates their belief to a weighted average of the beliefs of agents they pay attention to. For instance, agent 1's belief becomes:

$$
\begin{aligned}
x_1^1 &= 0.5 \cdot 0 + 0.3 \cdot 1 + 0.2 \cdot 0.2 \\
&= 0.34.
\end{aligned}
$$



*Greener means closer to the truth, redder is farther away.

THE DEGROOT MODEL: EXAMPLE

Take $N = \{1, \ldots, 6\}$ to be the set of agents. They are connected by the social network G on the right.

Each agent distributes a total weight of 1 across the agents in their neighborhood.

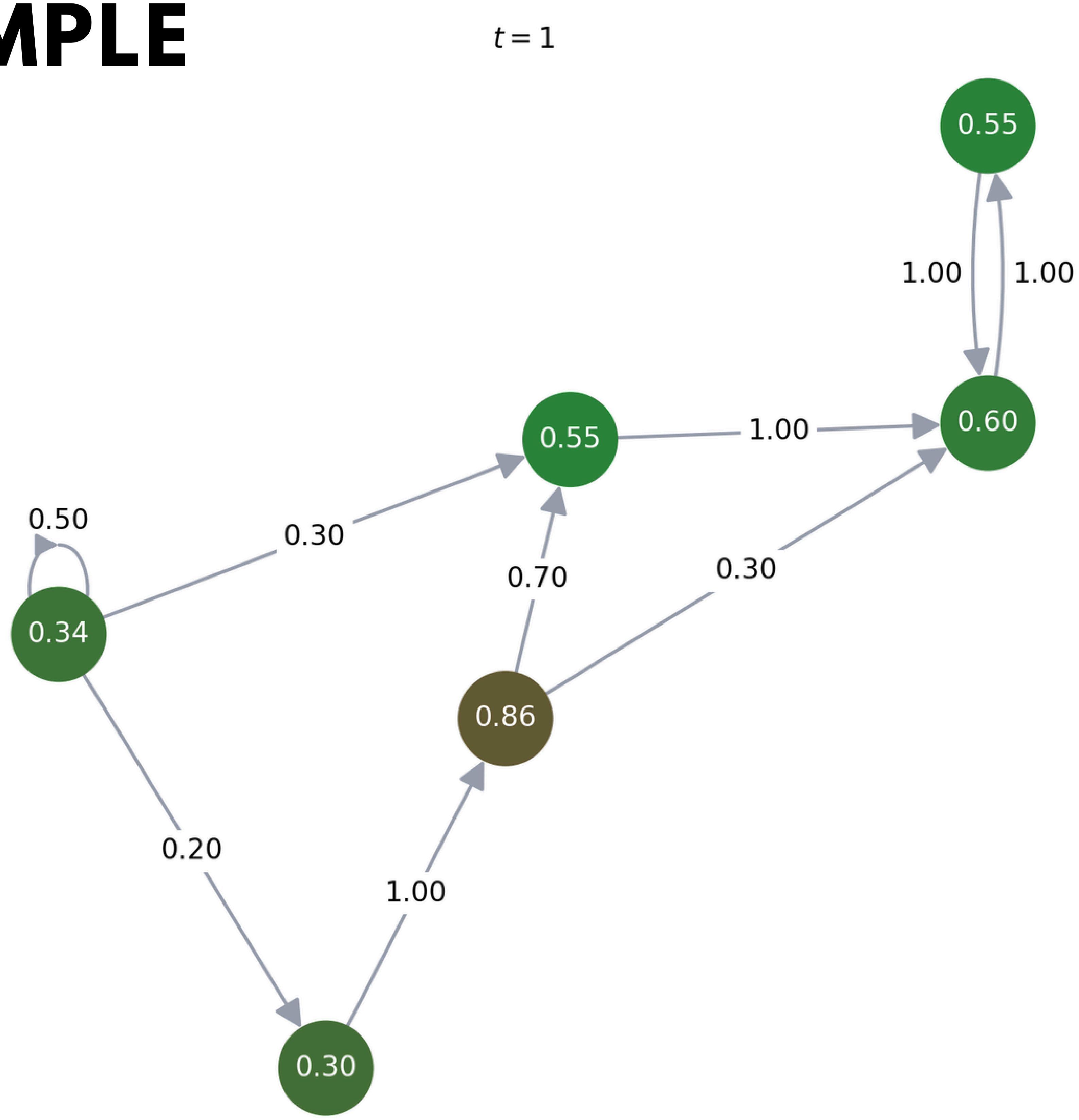
The true state is $\mu = 0.5$. Each agent starts with the initial belief shown on the right.*

Time starts at $t = 0$.

At $t = 1$, each agent updates their belief to a weighted average of the beliefs of agents they pay attention to. For instance, agent 1's belief becomes:

$$x_1^1 = 0.5 \cdot 0 + 0.3 \cdot 1 + 0.2 \cdot 0.2$$
$$= 0.34.$$

This keeps going for as long as we like...



*Greener means closer to the truth, redder is farther away.

THE DEGROOT MODEL: EXAMPLE

Take $N = \{1, \ldots, 6\}$ to be the set of agents. They are connected by the social network G on the right.

Each agent distributes a total weight of 1 across the agents in their neighborhood.

The true state is $\mu = 0.5$. Each agent starts with the initial belief shown on the right.*

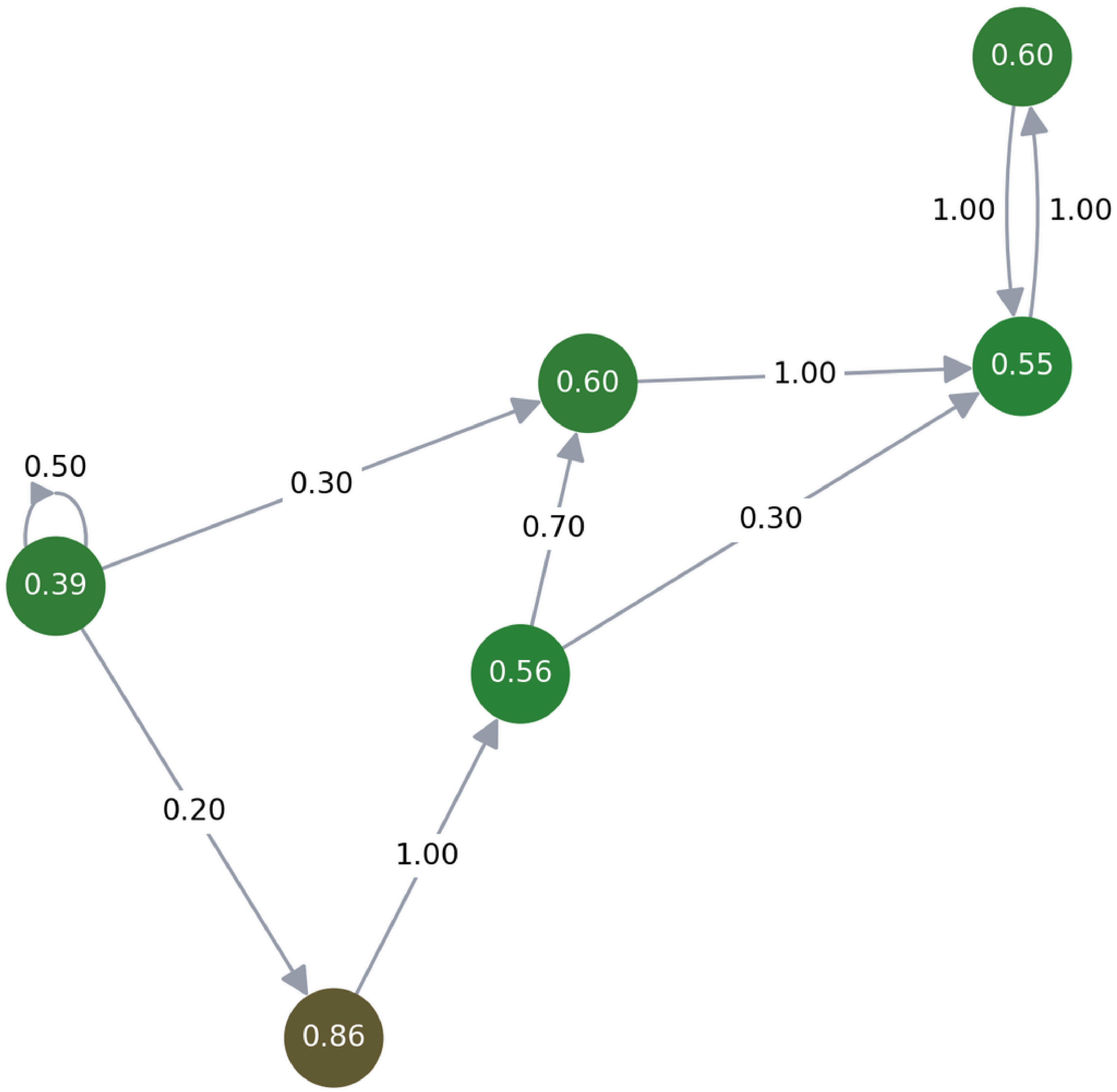
Time starts at $t = 0$.

At $t = 1$, each agent updates their belief to a weighted average of the beliefs of agents they pay attention to. For instance, agent 1's belief becomes:

$$\begin{aligned} x_1^1 &= 0.5 \cdot 0 + 0.3 \cdot 1 + 0.2 \cdot 0.2 \\ &= 0.34. \end{aligned}$$

This keeps going for as long as we like…

$t = 2$



*Greener means closer to the truth, redder is farther away.

THE DEGROOT MODEL: EXAMPLE

Take $N = \{1, \ldots, 6\}$ to be the set of agents. They are connected by the social network G on the right.

Each agent distributes a total weight of 1 across the agents in their neighborhood.

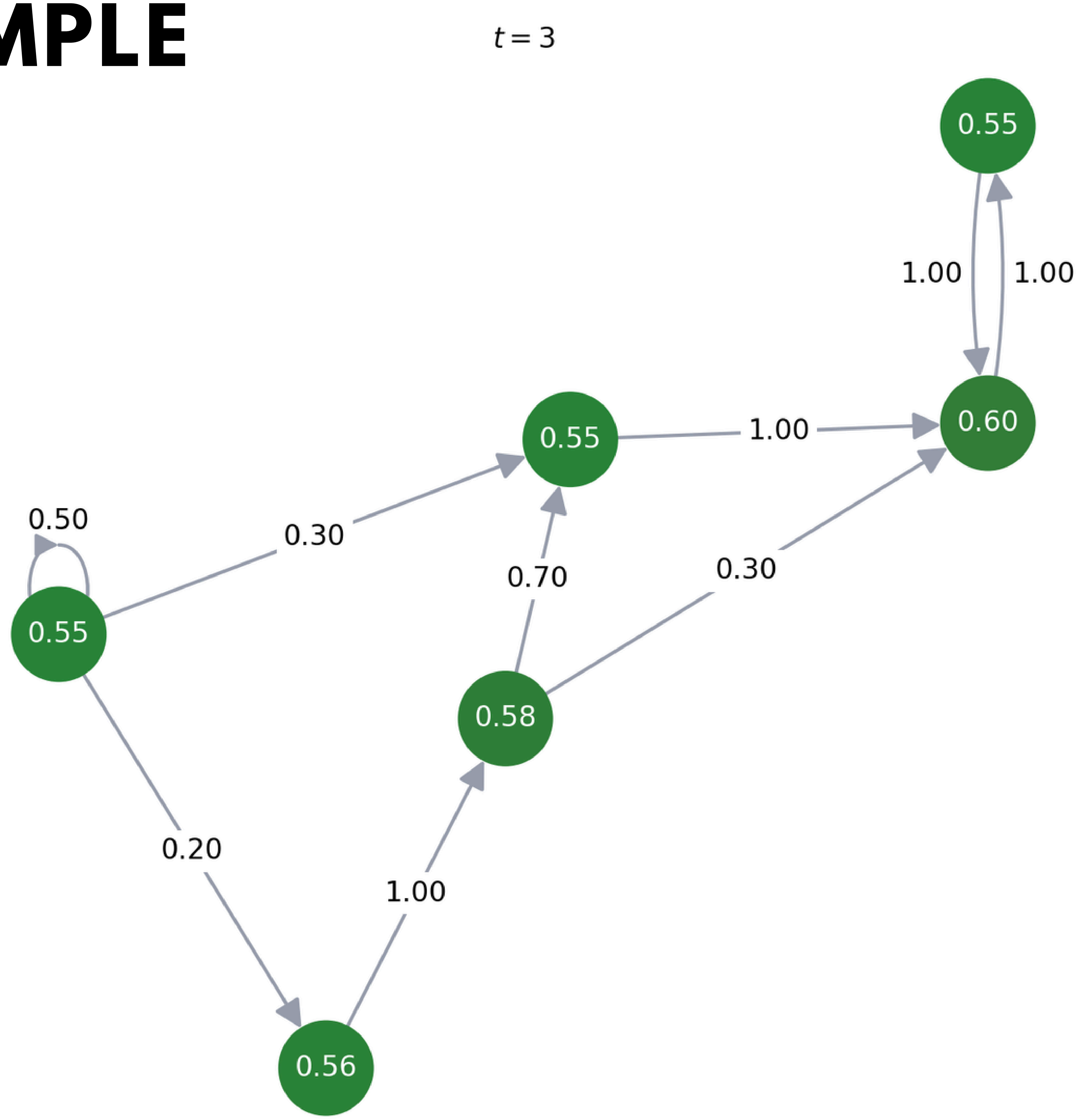
The true state is $\mu = 0.5$. Each agent starts with the initial belief shown on the right.*

Time starts at $t = 0$.

At $t = 1$, each agent updates their belief to a weighted average of the beliefs of agents they pay attention to. For instance, agent 1's belief becomes:

$$x_1^1 = 0.5 \cdot 0 + 0.3 \cdot 1 + 0.2 \cdot 0.2$$
$$= 0.34.$$

This keeps going for as long as we like...



*Greener means closer to the truth, redder is farther away.

THE DEGROOT MODEL: EXAMPLE

Take $N = \{1, \ldots, 6\}$ to be the set of agents. They are connected by the social network G on the right.

Each agent distributes a total weight of 1 across the agents in their neighborhood.

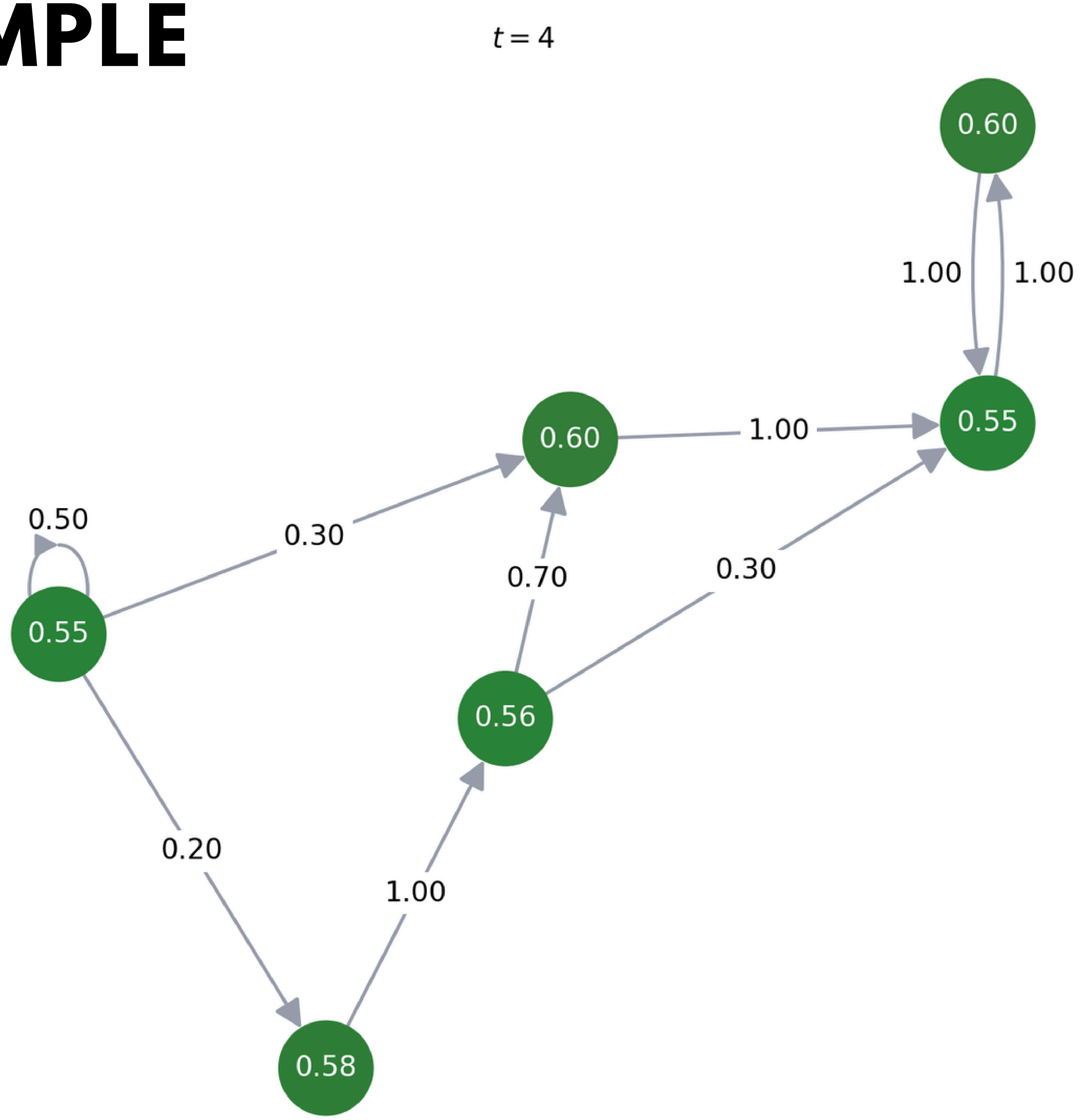
The true state is $\mu = 0.5$. Each agent starts with the initial belief shown on the right.*

Time starts at $t = 0$.

At $t = 1$, each agent updates their belief to a weighted average of the beliefs of agents they pay attention to. For instance, agent 1's belief becomes:

$$x_1^1 = 0.5 \cdot 0 + 0.3 \cdot 1 + 0.2 \cdot 0.2 \\ = 0.34.$$

This keeps going for as long as we like…



*Greener means closer to the truth, redder is farther away.

THE DEGROOT MODEL: EXAMPLE

Take $N = \{1, \ldots, 6\}$ to be the set of agents. They are connected by the social network G on the right.

Each agent distributes a total weight of 1 across the agents in their neighborhood.

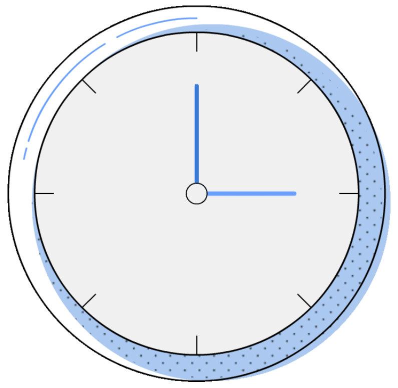
The true state is $\mu = 0.5$. Each agent starts with the initial belief shown on the right.*

Time starts at $t = 0$.

At $t = 1$, each agent updates their belief to a weighted average of the beliefs of agents they pay attention to. For instance, agent 1's belief becomes:

$$\begin{aligned} x_1^1 &= 0.5 \cdot 0 + 0.3 \cdot 1 + 0.2 \cdot 0.2 \\ &= 0.34. \end{aligned}$$

This keeps going for as long as we like...



THE DEGROOT MODEL: EXAMPLE

Take $N = \{1, \ldots, 6\}$ to be the set of agents. They are connected by the social network G on the right.

Each agent distributes a total weight of 1 across the agents in their neighborhood.

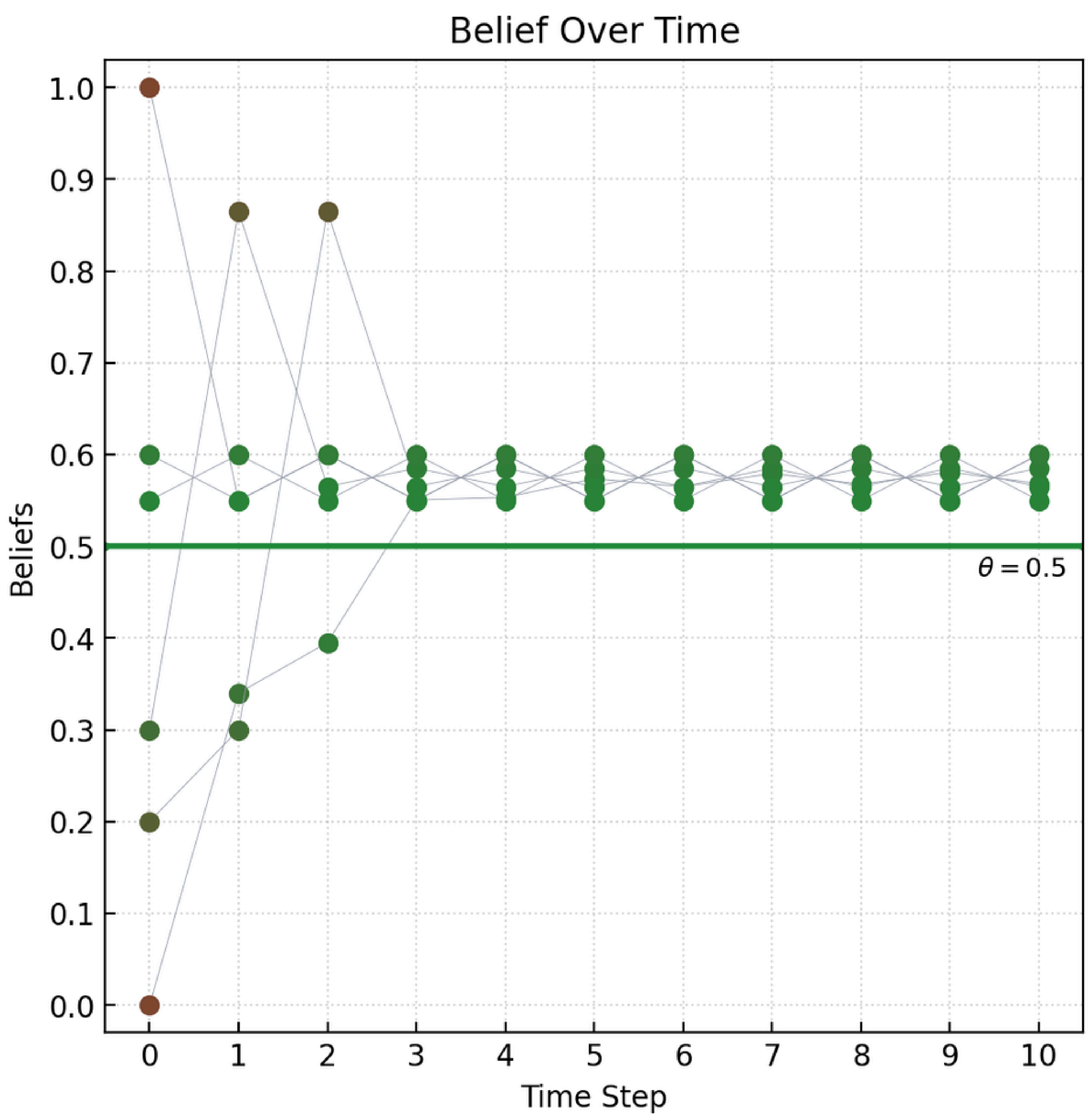
The true state is $\mu = 0.5$. Each agent starts with the initial belief shown on the right.*

Time starts at $t = 0$.

At $t = 1$, each agent updates their belief to a weighted average of the beliefs of agents they pay attention to. For instance, agent 1's belief becomes:

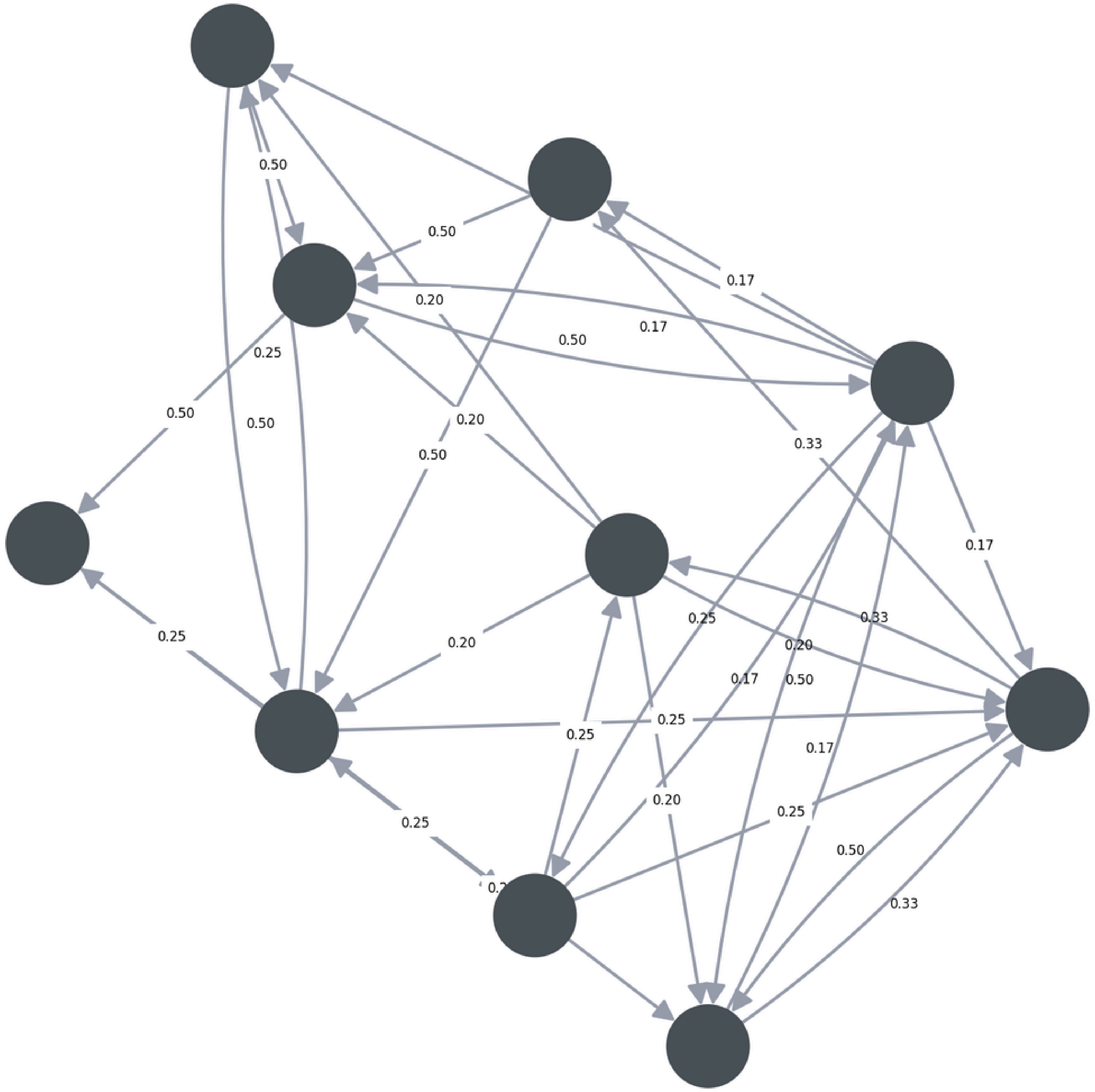
$$
\begin{aligned}
x_1^1 &= 0.5 \cdot 0 + 0.3 \cdot 1 + 0.2 \cdot 0.2 \\
&= 0.34.
\end{aligned}
$$

This keeps going for as long as we like…



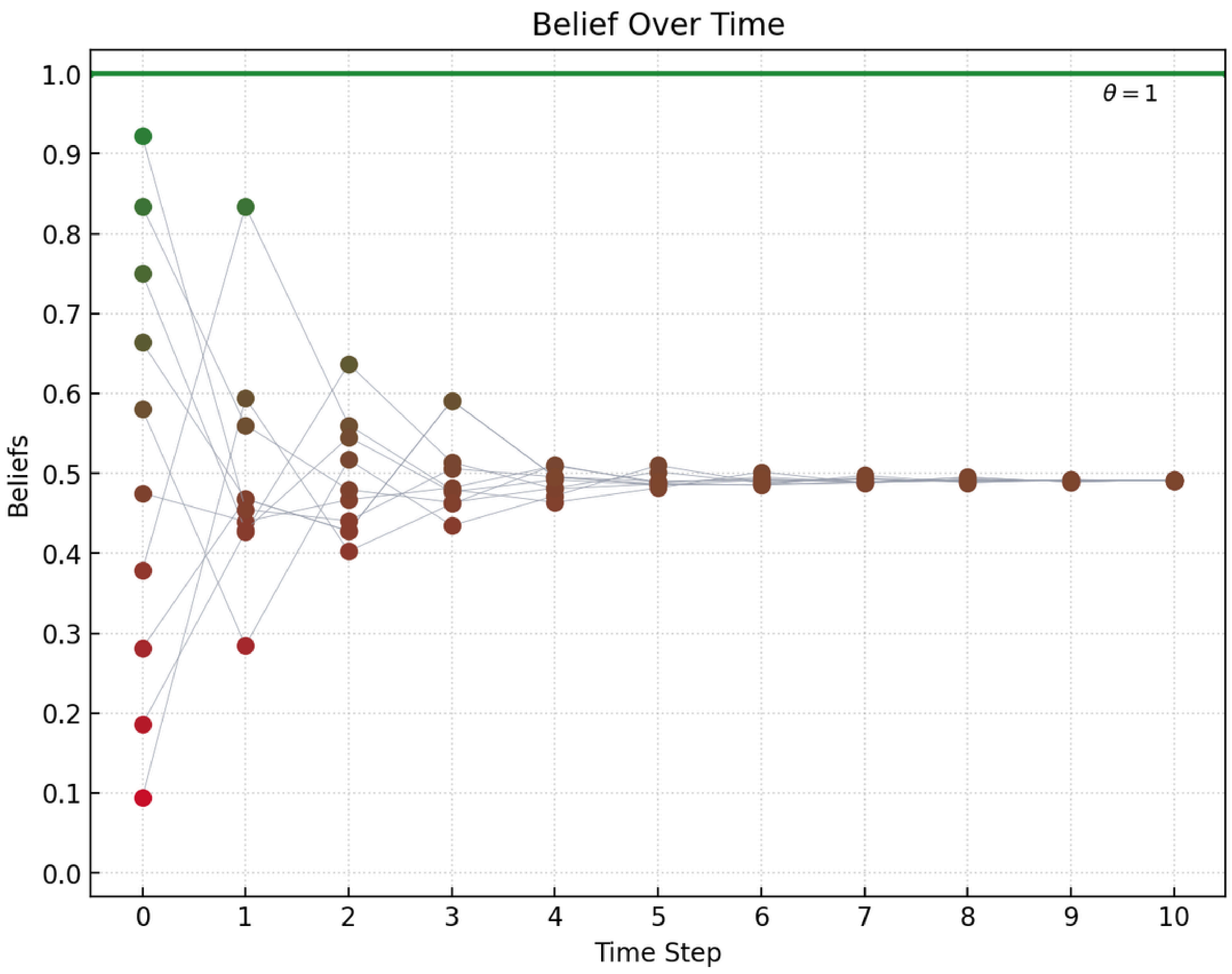
RANDOM NETWORKS SIMULATION

Take a G(n, p) Erdős–Rényi random network with n = 10 nodes and p = 0.3.



RANDOM NETWORKS SIMULATION

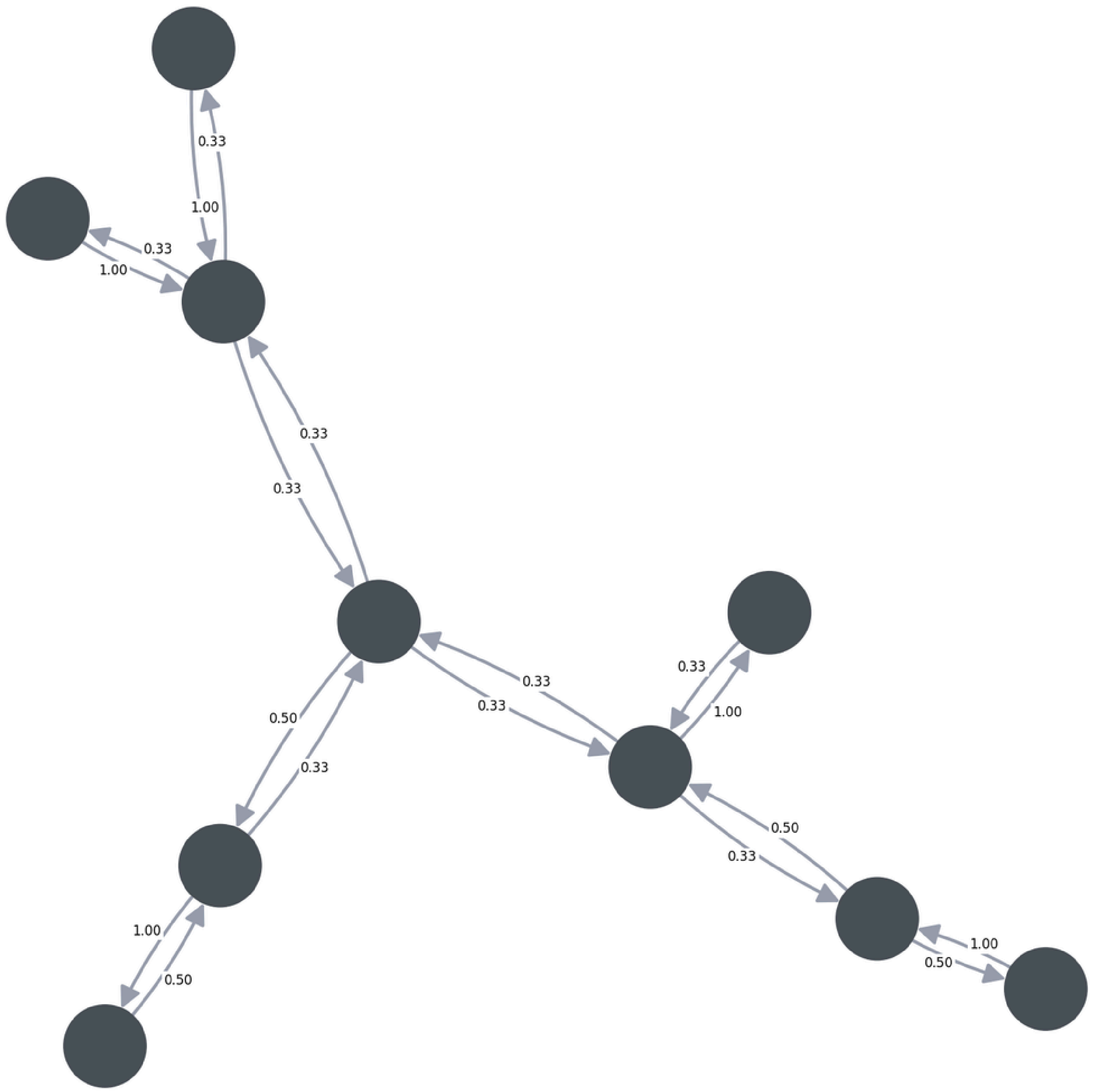
Take a G(n, p) Erdős–Rényi random network with n = 10 nodes and p = 0.3.



Results are averages over a batch of 100 sets of uniformly random initial beliefs.

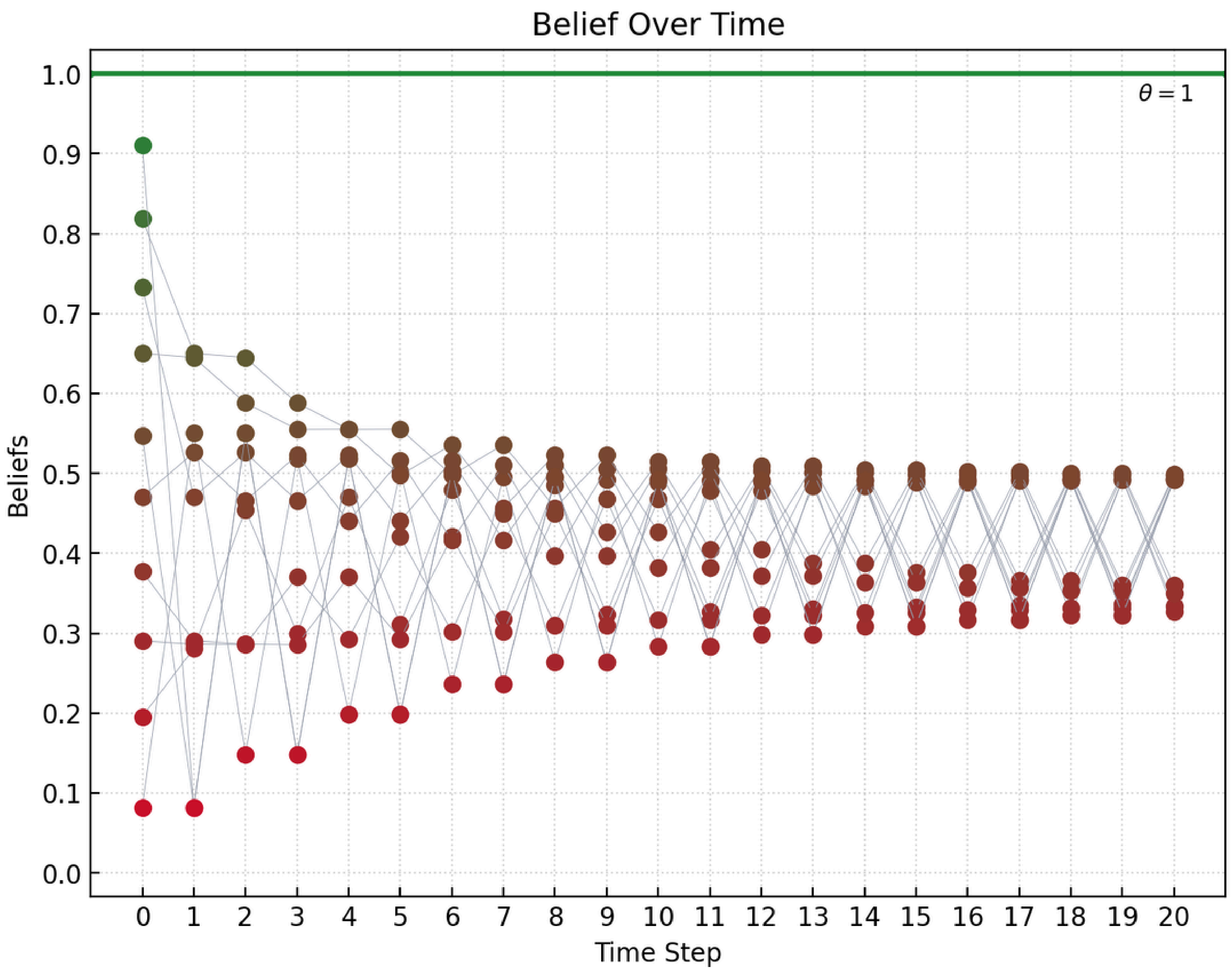
SCALE FREE NETWORKS SIMULATION

Take a Barabási–Albert graph with n = 10 nodes.



SCALE FREE NETWORKS SIMULATION

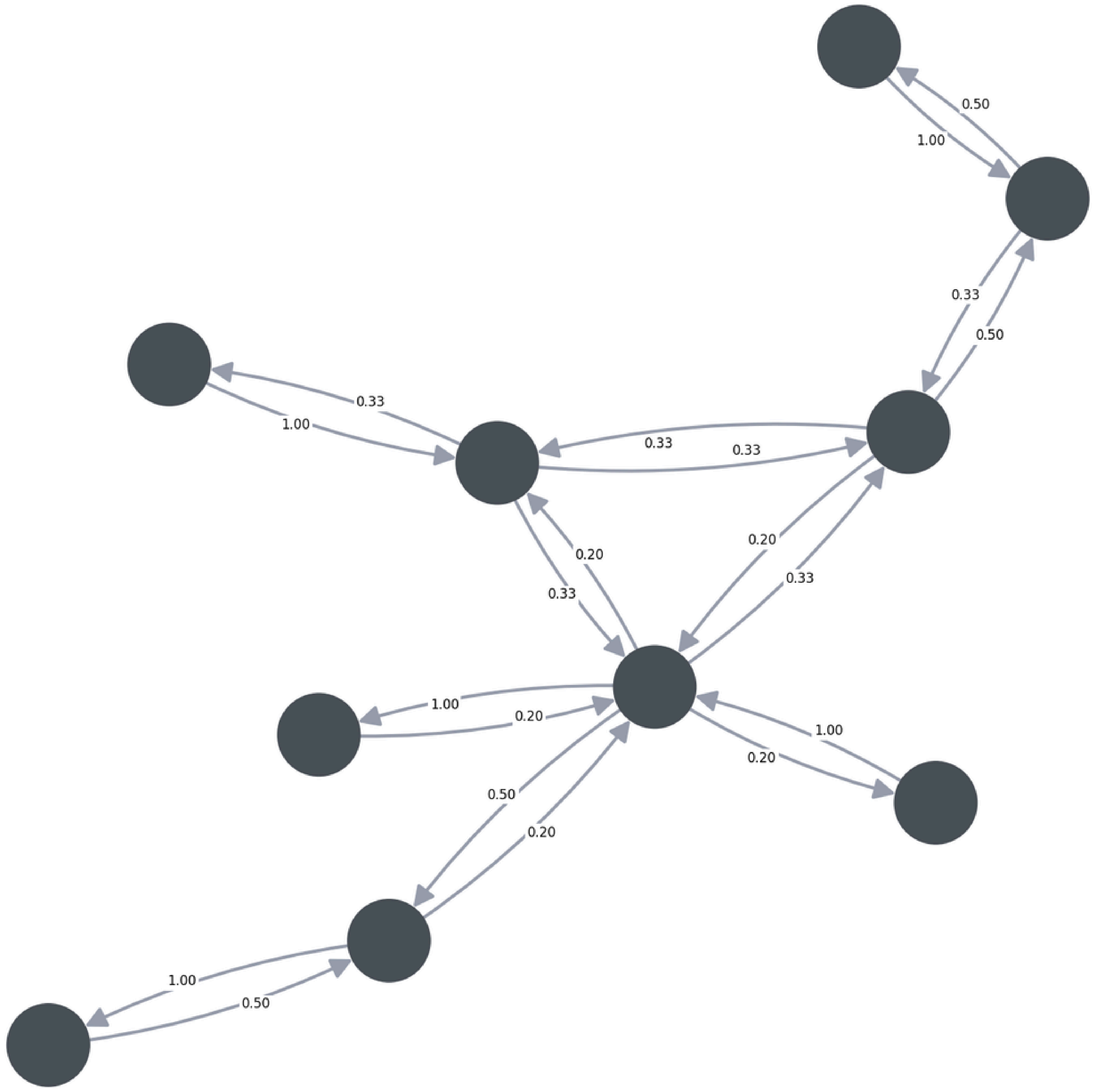
Take a Barabási–Albert graph with n = 10 nodes.



Results are averages over a batch of 100 sets of uniformly random initial beliefs.

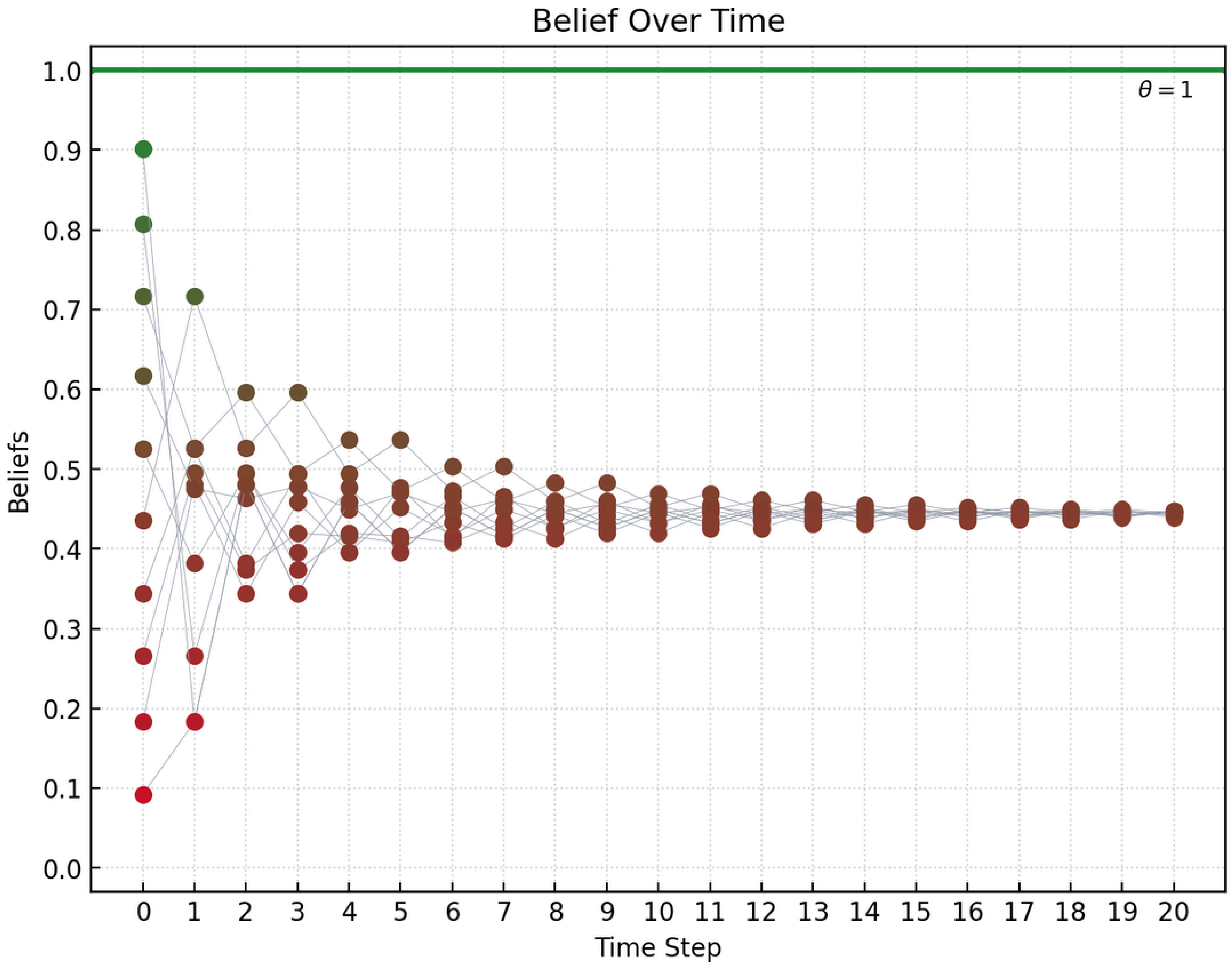
SMALL-WORLD NETWORKS SIMULATION

Take a Watts-Strogatz graph with n = 10 nodes.



SMALL-WORLD NETWORKS SIMULATION

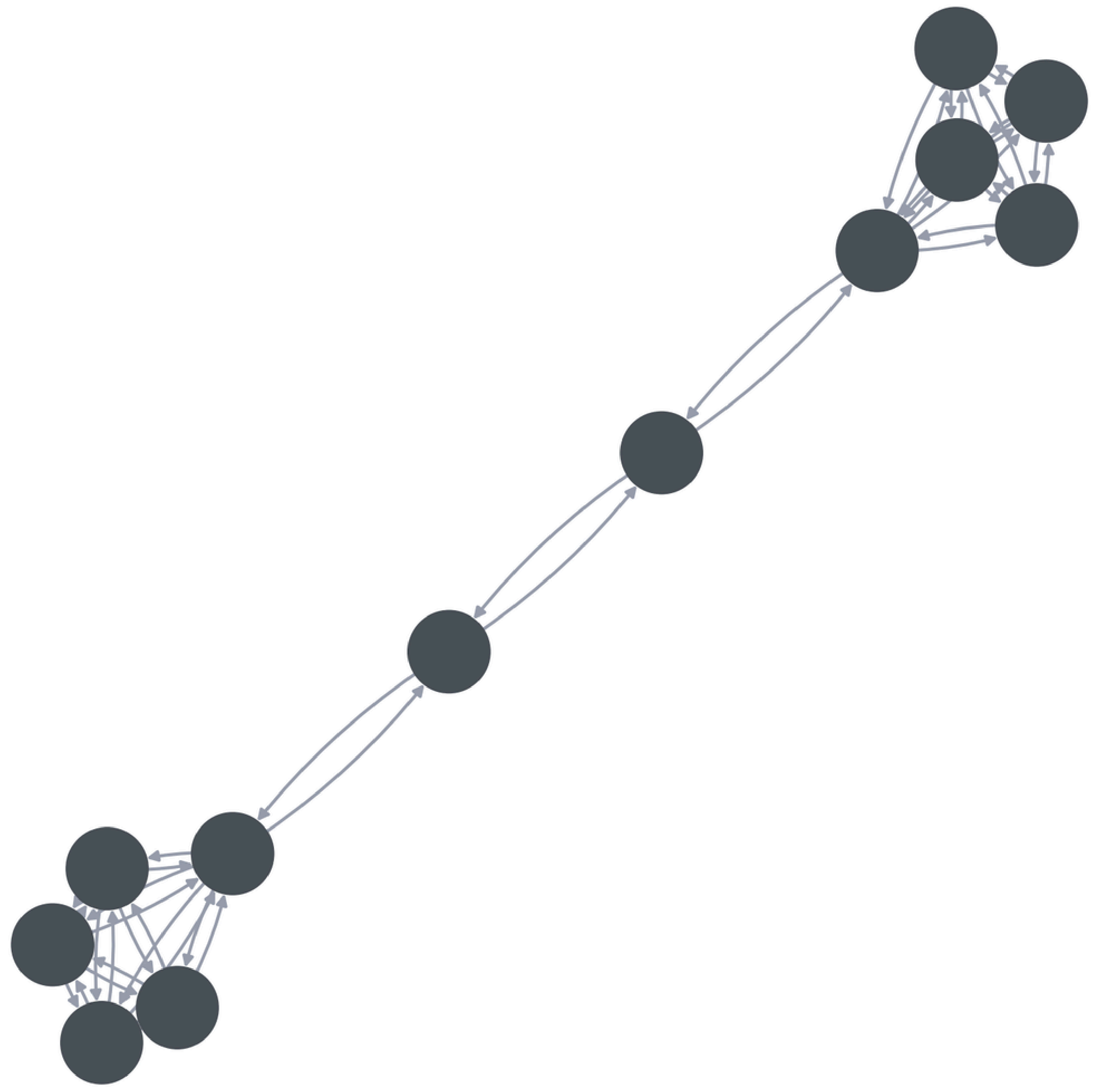
Take a Watts-Strogatz graph with n = 10 nodes.



Results are averages over a batch of 100 sets of uniformly random initial beliefs.

BARBELL GRAPH SIMULATION

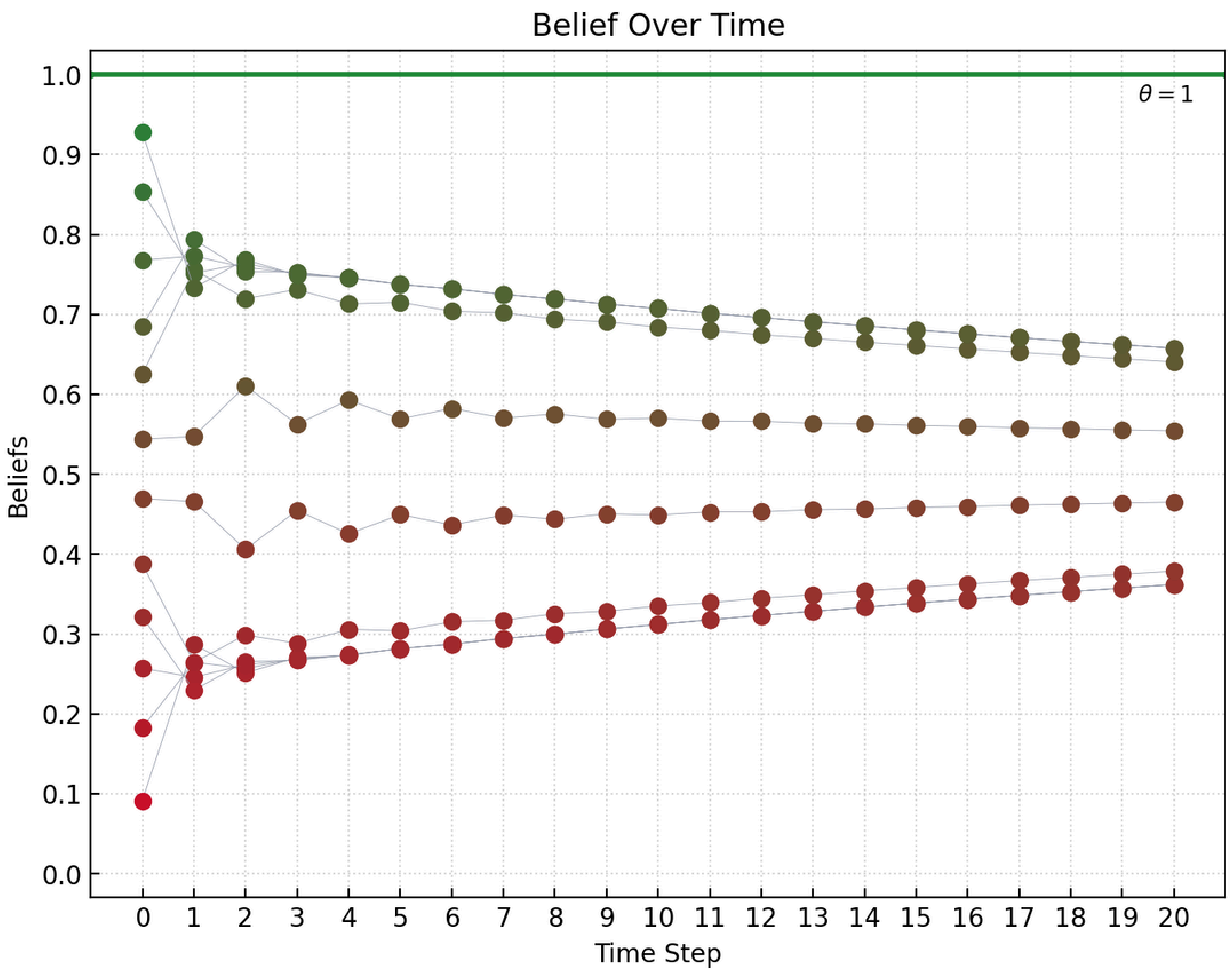
Take a ‘barbell’ graph.



BARBELL GRAPH SIMULATION

Take a ‘barbell’ graph.

Results seem consistent with a polarized society.



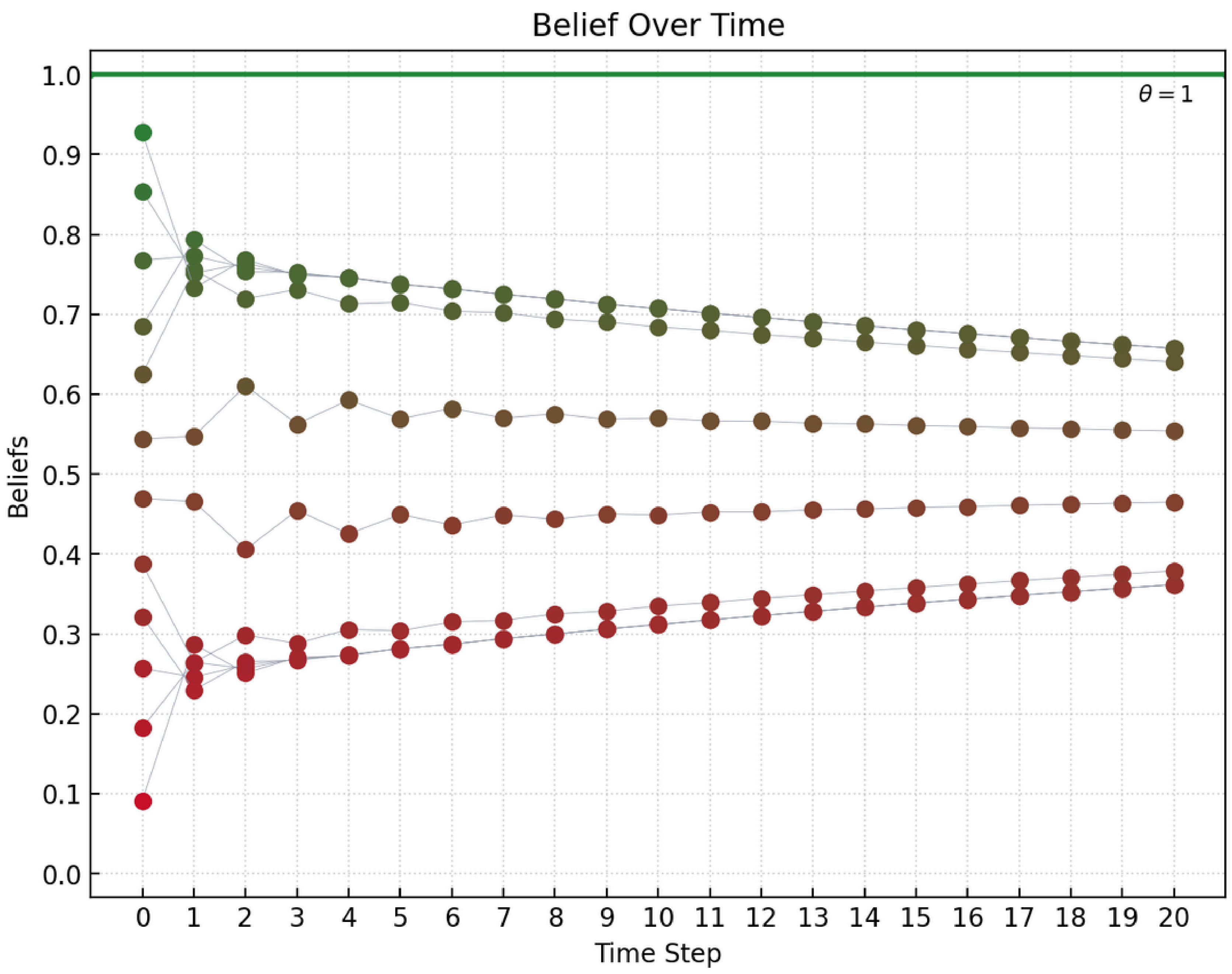
Results are averages over a batch of 100 sets of uniformly random initial beliefs.

BARBELL GRAPH SIMULATION

Take a ‘barbell’ graph.

Results seem consistent with a polarized society.

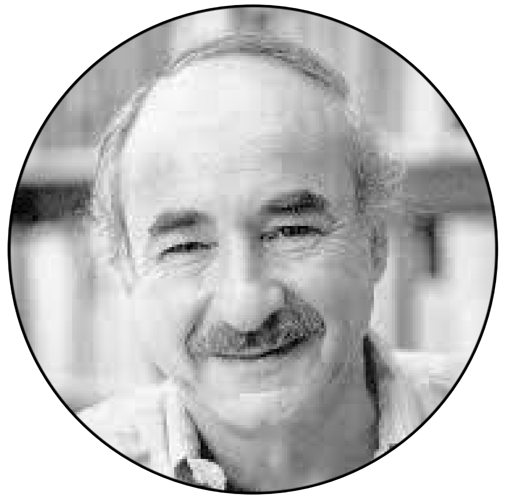
Though variance in beliefs still seems to go down…



Results are averages over a batch of 100 sets of uniformly random initial beliefs.

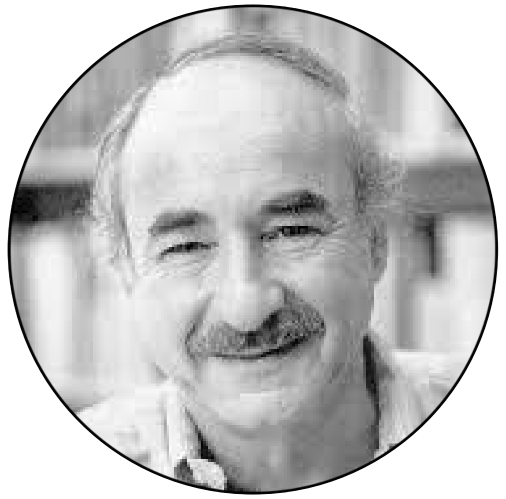
Do individual beliefs ever *converge*, i.e., stop changing?

Do individual beliefs ever *converge*, i.e., stop changing? And if they do, do they converge to the same value, i.e., a *consensus*?



MORRIS DEGROOT

Yes!



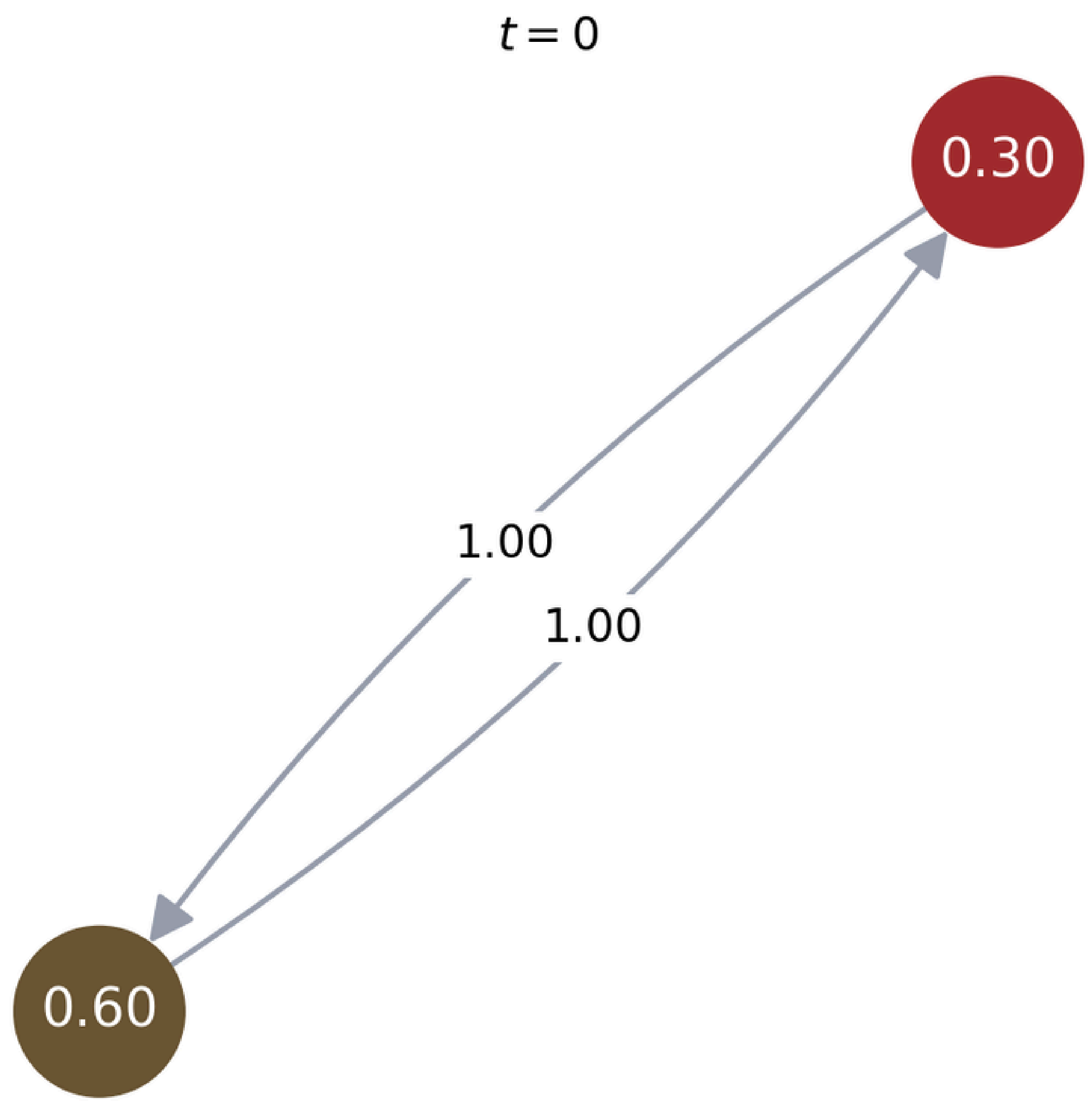
MORRIS DEGROOT

Yes!

Under certain conditions...

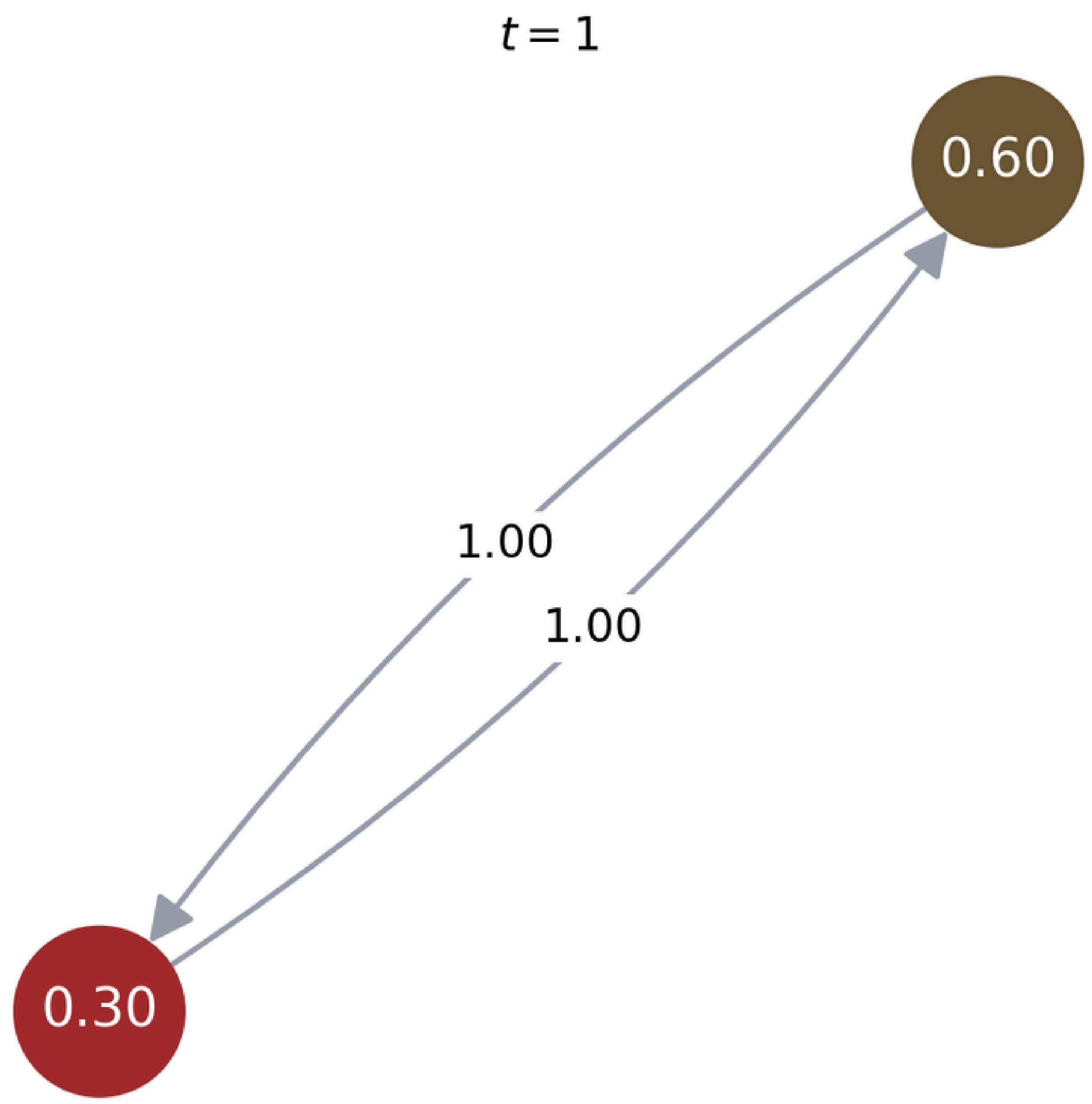
CYCLES

Cycles are bad news.



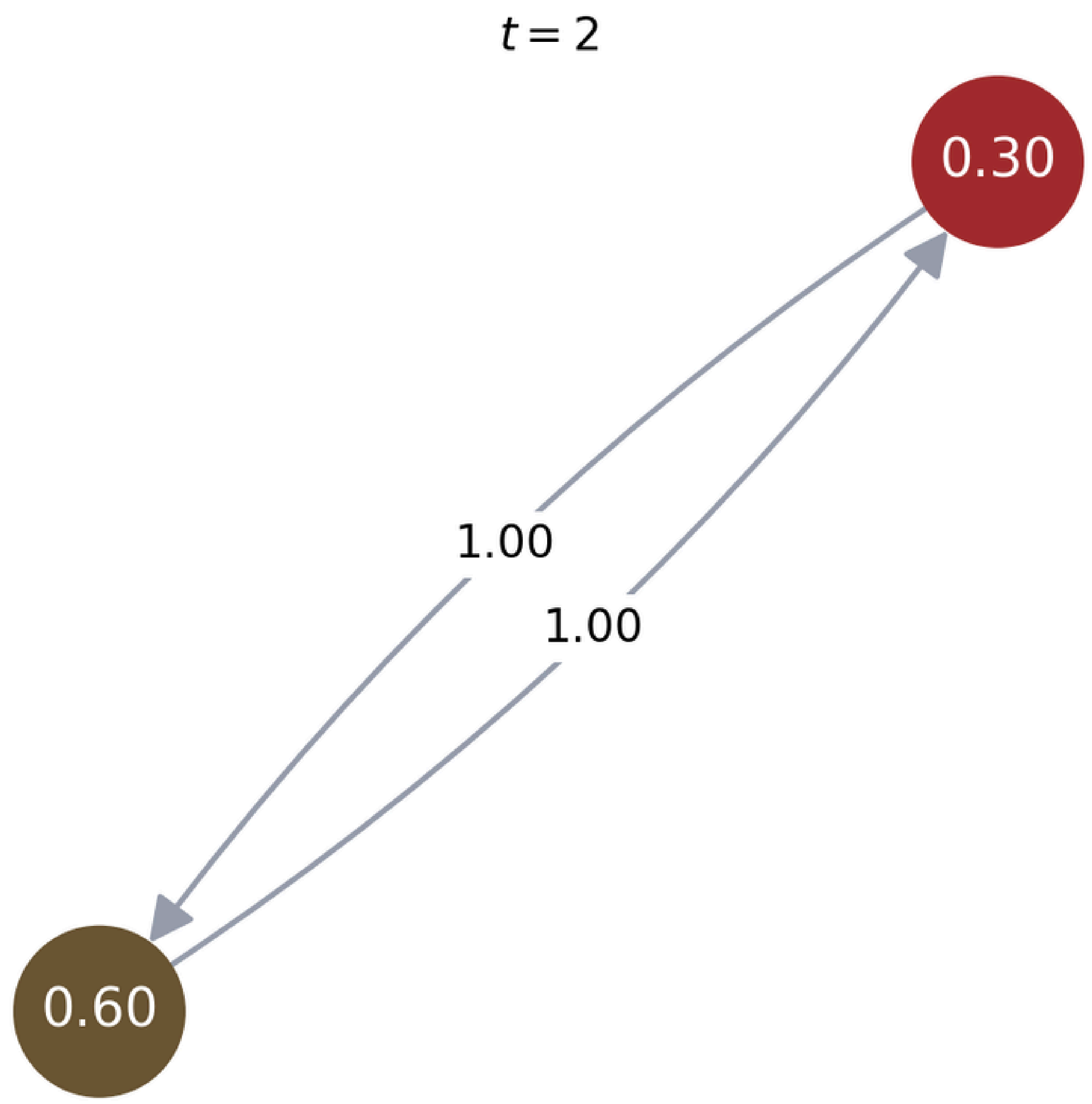
CYCLES

Cycles are bad news.



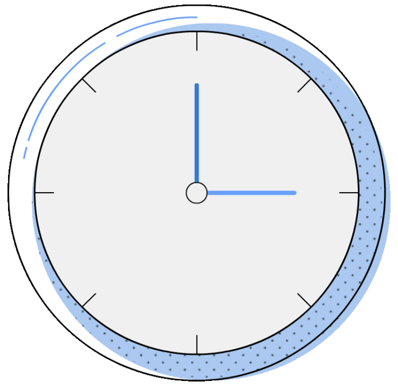
CYCLES

Cycles are bad news.



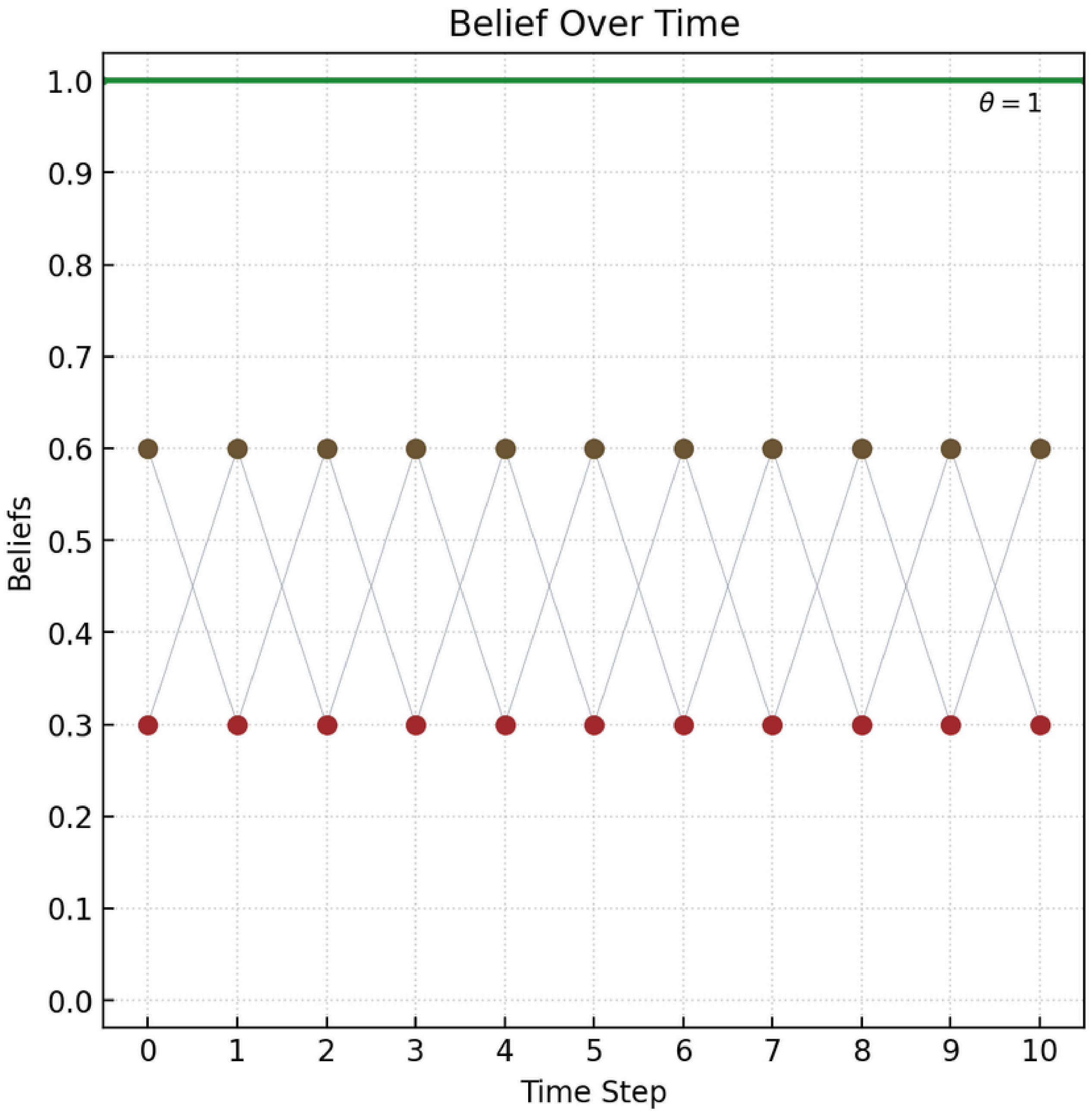
CYCLES

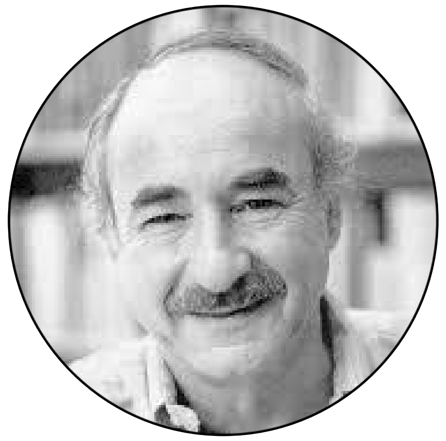
Cycles are bad news.



CYCLES

Cycles are bad news.



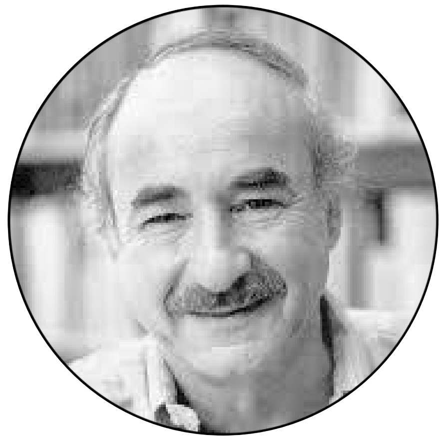


MORRIS DEGROOT

Ok, then let’s assume there aren’t any (bad) cycles.

DEFINITION (APERIODICITY)

A network is *aperiodic* if the greatest common divisor of any two cycle lengths is 1.



MORRIS DEGROOT

It’s fine to have cycles of length 2, 3, 4.

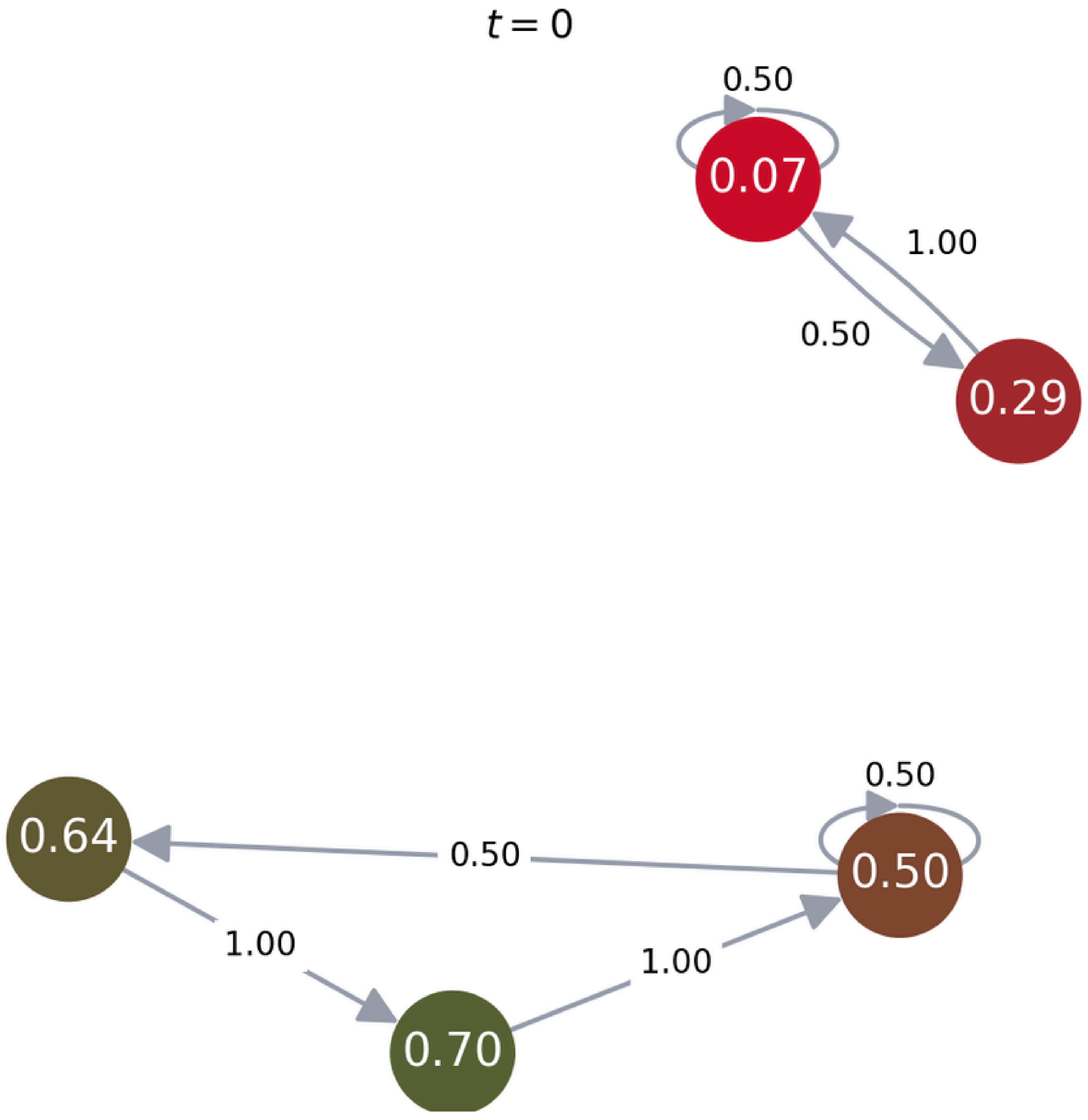
But not cycles of length 2 and 4.

Or 3 and 6.

An easy way of a making a network aperiodic is by adding a self loop.

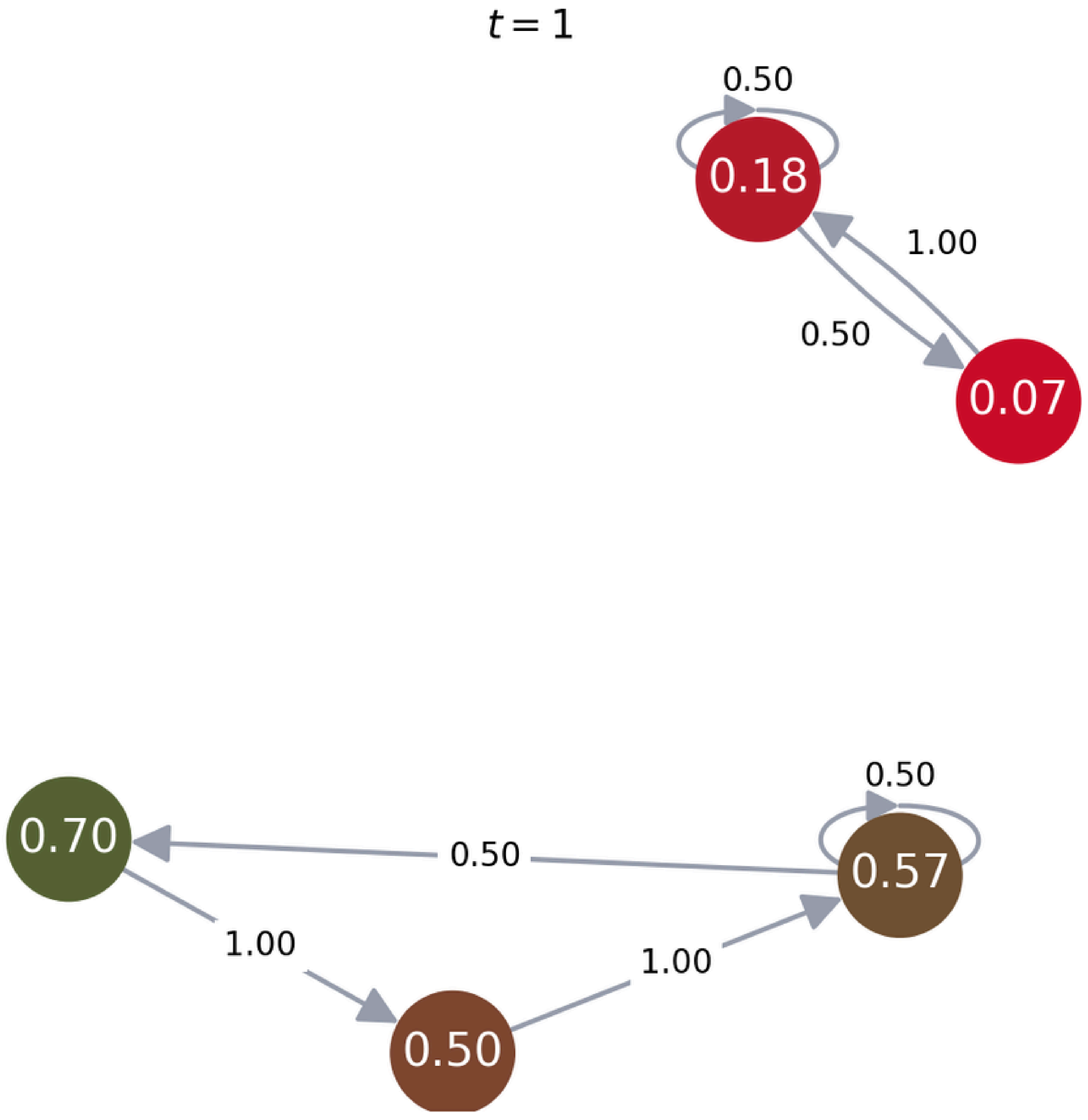
DISCONNECTED COMPONENTS

Disconnected components
are also bad news.



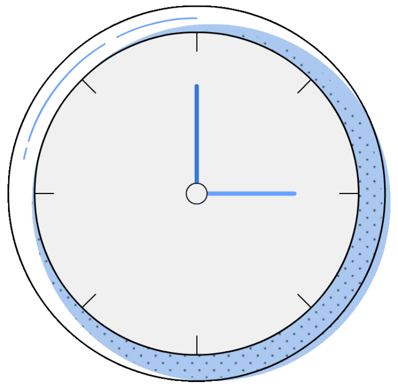
DISCONNECTED COMPONENTS

Disconnected components are also bad news.



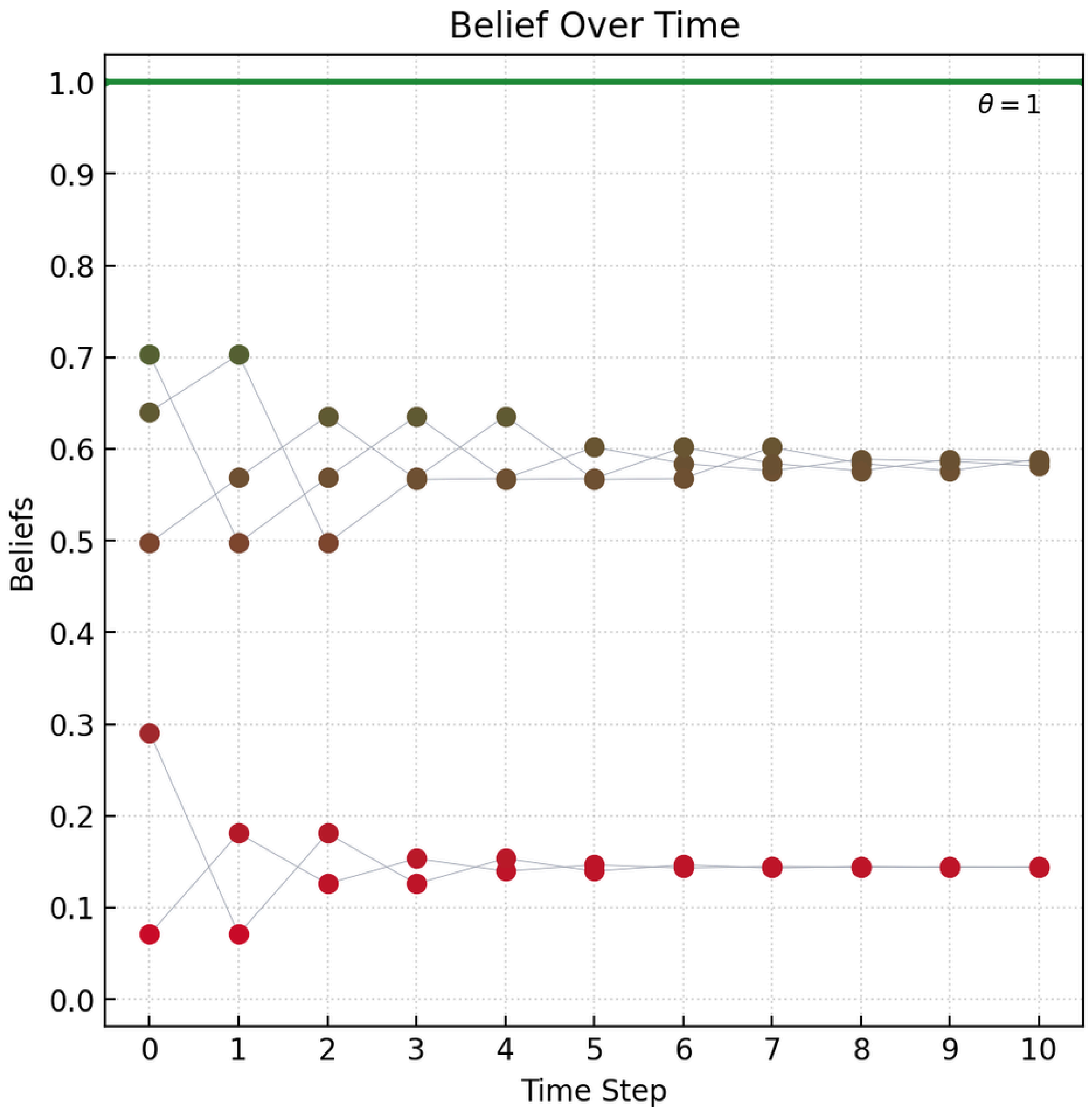
DISCONNECTED COMPONENTS

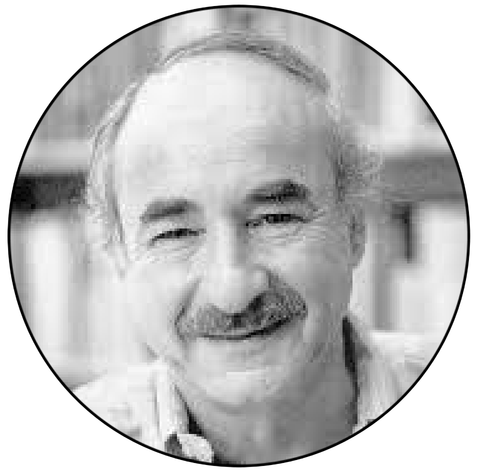
Disconnected components
are also bad news.



DISCONNECTED COMPONENTS

Disconnected components are also bad news.



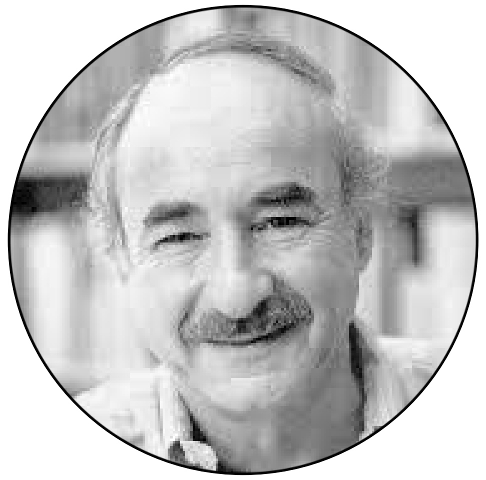


MORRIS DEGROOT

Ok, let’s assume no isolated components.

DEFINITION (STRONG CONNECTEDNESS)

A network is *strongly connected* if there is a path from any node to any other node.



MORRIS DEGROOT

Aperiodicity and strong connectedness do the trick.

THEOREM (DEGROOT, 1974)

If the social network is strongly connected and aperiodic, then the agents' opinions converge to a common value $\tilde{x} \in [0, 1]$, called the *consensus belief*:

$$\lim_{t \to \infty} x_i^t = \tilde{x},$$

for all agents i.

DeGroot, M. H. (1974). Reaching a Consensus. *Journal of the American Statistical Association*, 69(345), 118–121.

Nice! But what needs to happen for agents in the DeGroot model to arrive at a consensus that is also *correct*?

BENJAMIN GOLUB

We want to speak of wise *networks*.

BENJAMIN GOLUB

We want to speak of wise *networks*.

MATTHEW O. JACKSON

As with the Condorcet Jury Theorem, this is a limit condition as the network grows larger and larger.

Golub, B., & Jackson, M. O. (2010). Naïve Learning in Social Networks and the Wisdom of Crowds. *American Economic Journal: Microeconomics*, 2(1), 112–149.

DEFINITION (WISE NETWORKS)
We write G_n for a network with n vertices.

Golub, B., & Jackson, M. O. (2010). Naïve Learning in Social Networks and the Wisdom of Crowds. *American Economic Journal: Microeconomics*, 2(1), 112–149.

DEFINITION (WISE NETWORKS)
We write G_n for a network with n vertices.

A sequence G_1, G_2, …, of networks of increasing size is *wise* if each network G_i admits a consensus belief, and the consensus belief approaches the true state μ asymptotically, as n goes to infinity:

$$\lim_{n\to\infty}\left(\lim_{t\to\infty} x_i^t, \text{ for every } i \text{ in } G_n\right) = \mu.$$

Golub, B., & Jackson, M. O. (2010). Naïve Learning in Social Networks and the Wisdom of Crowds. *American Economic Journal: Microeconomics*, 2(1), 112–149.

At the same time, this is a much stronger condition than in the Condorcet Jury Theorem (CJT).

At the same time, this is a much stronger condition than in the Condorcet Jury Theorem (CJT). In the CJT we do not assume there is a consensus…

At the same time, this is a much stronger condition than in the Condorcet Jury Theorem (CJT). In the CJT we do not assume there is a consensus… we look at the *average* belief.

BENJAMIN GOLUB

The consensus belief is interesting, when it exists, because there turns out to be a really cool way of thinking of it.

BENJAMIN GOLUB

The consensus belief is interesting, when it exists, because there turns out to be a really cool way of thinking of it.

MATTHEW O. JACKSON

And it involves the centrality of the nodes!

DEFINITION (WEIGHT MATRIX)

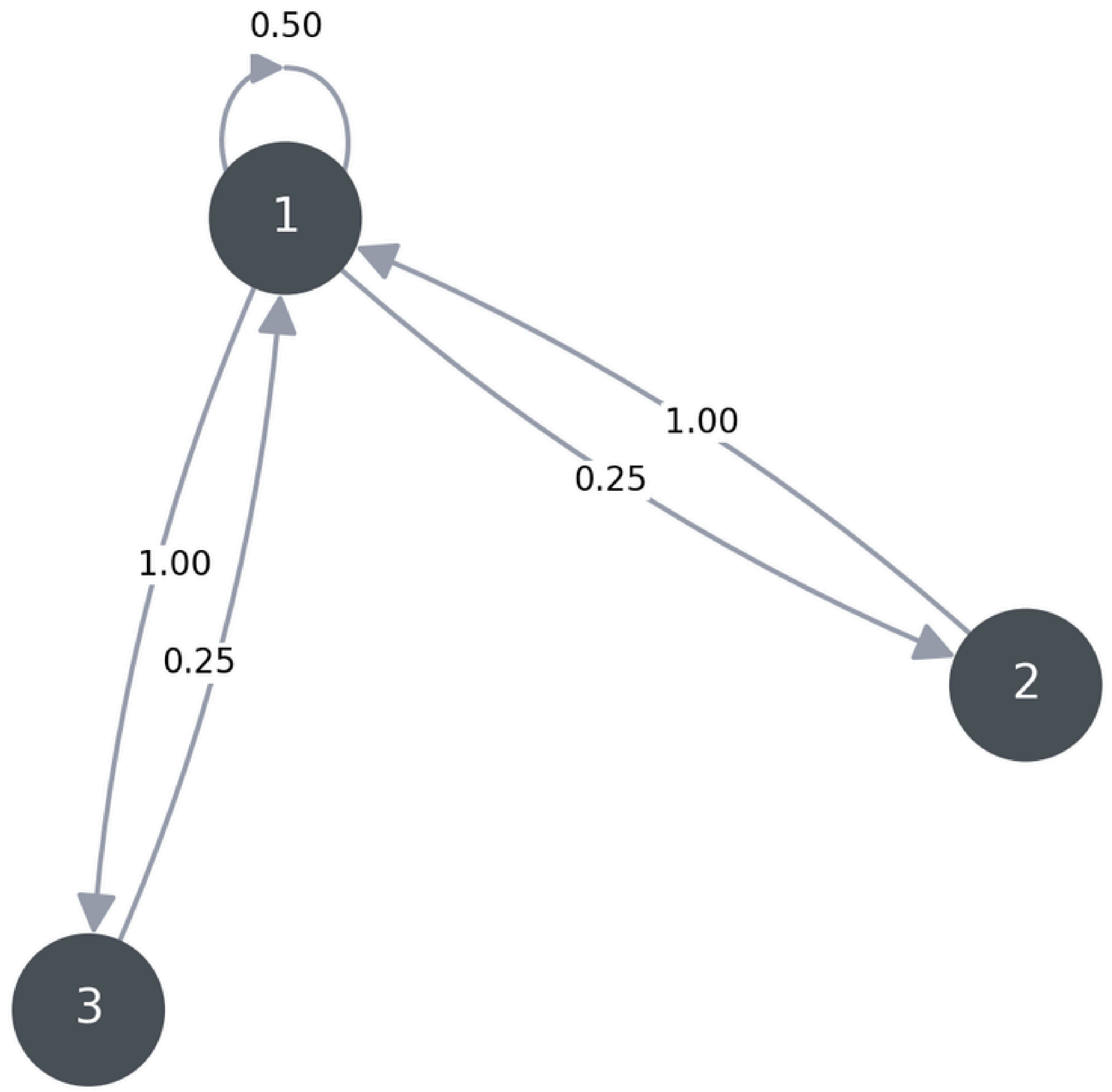
The *weight matrix* of network G is a matrix $W = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{bmatrix}$, where:

$$w_{ij} = \begin{cases} \text{weight that agent } i \text{ places on agent } j\text{'s opinion,} & \text{if } (i,j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

THE LINEAR ALGEBRA OF CONSENSUS

Consider the graph on the right. The weight matrix is:

$$W = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

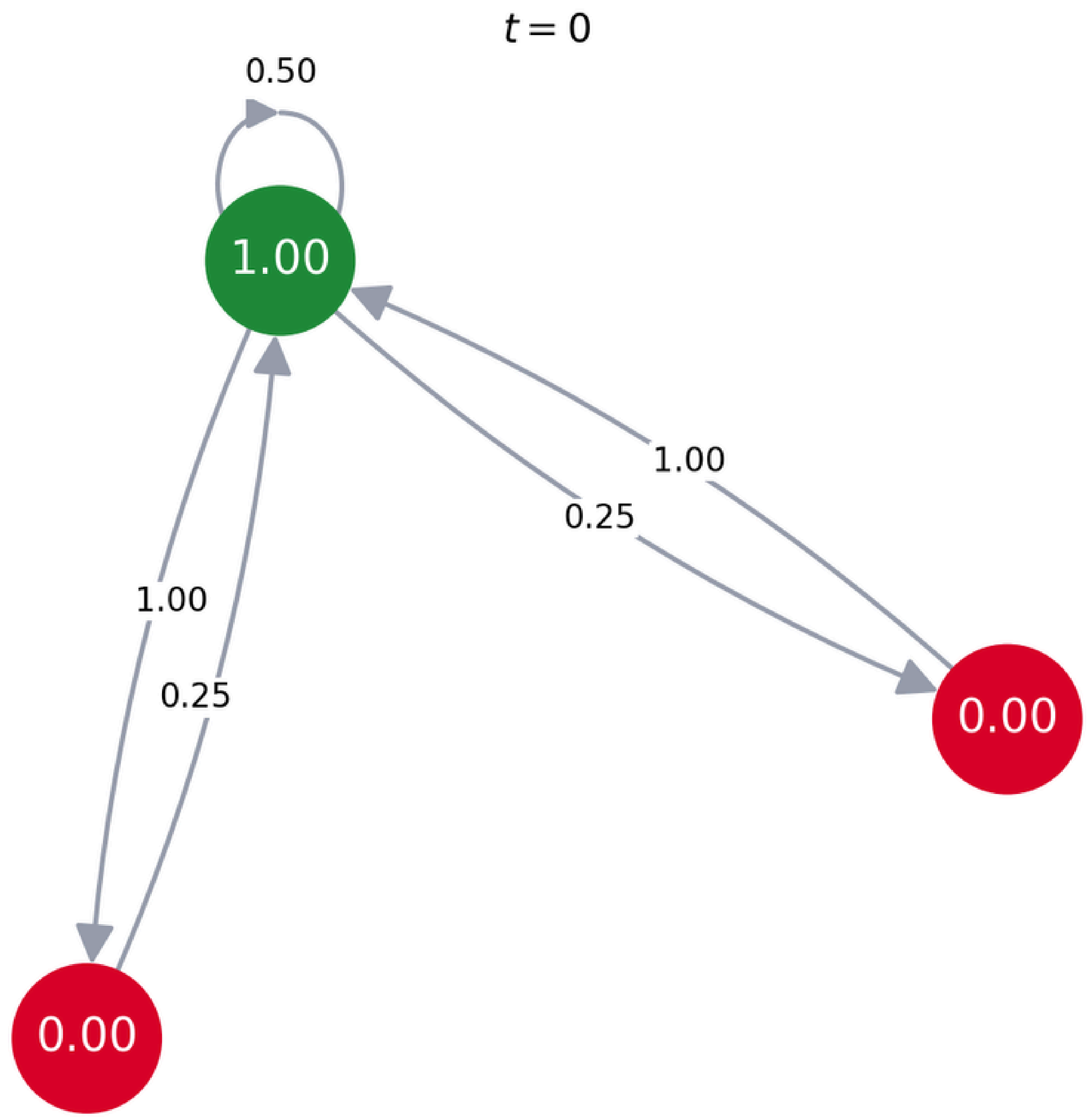


THE LINEAR ALGEBRA OF CONSENSUS

Consider the graph on the right. The weight matrix is:

$$W = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Take initial beliefs to be $x_1^0 = 1$, $x_2^0 = 0$, and $x_3^0 = 0$.



THE LINEAR ALGEBRA OF CONSENSUS

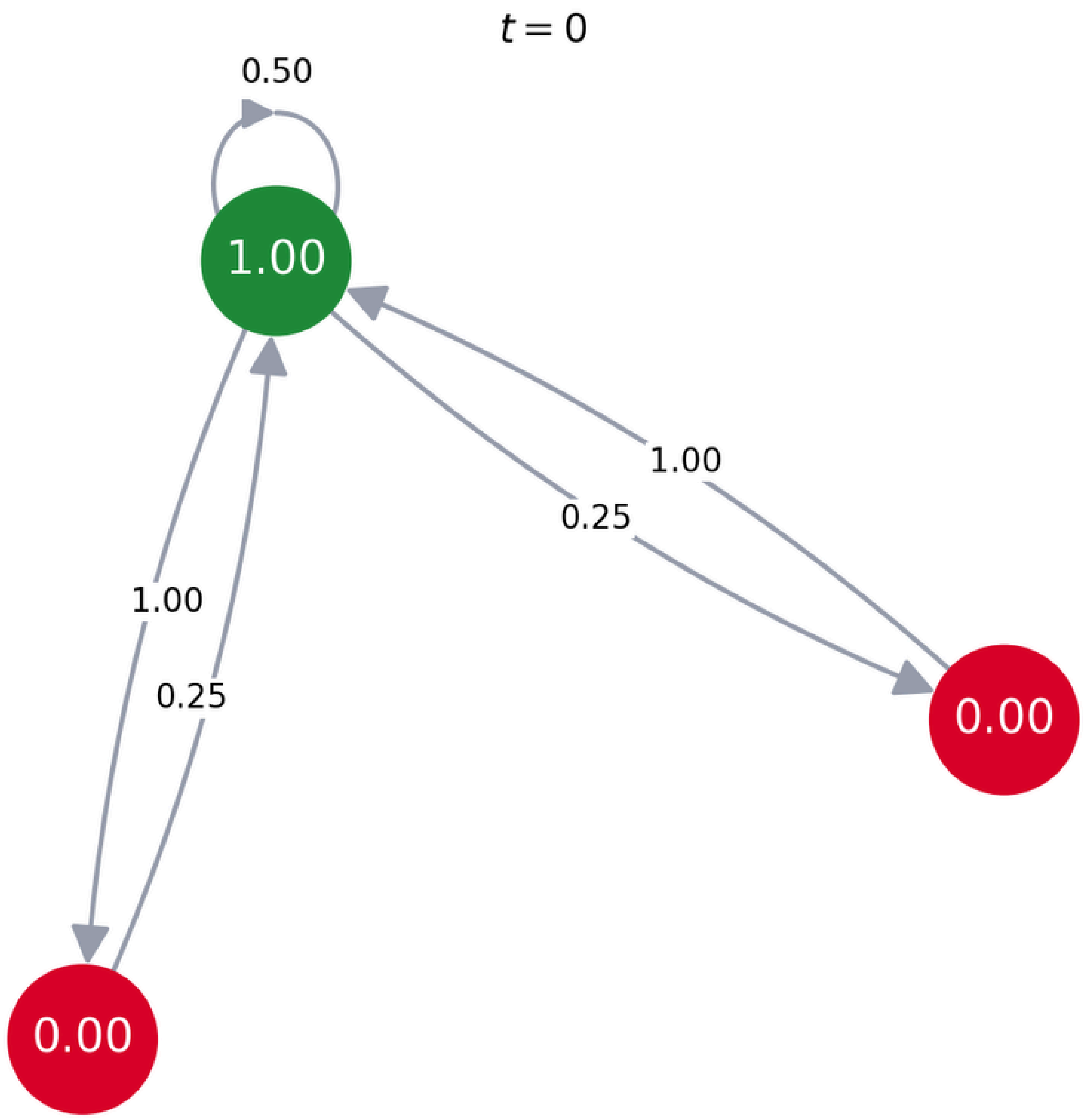
Consider the graph on the right. The weight matrix is:

$$W = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Take initial beliefs to be $x_1^0 = 1$, $x_2^0 = 0$, and $x_3^0 = 0$.

Beliefs at time $t = 1$ are:

$$\begin{aligned} \boldsymbol{x}^1 &= W \cdot \boldsymbol{x}^0 \\ &= \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.5 \cdot 1 + 0.25 \cdot 0 + 0.25 \cdot 0 \\ 1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 \\ 1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.5 \\ 1 \\ 1 \end{bmatrix}. \end{aligned}$$



THE LINEAR ALGEBRA OF CONSENSUS

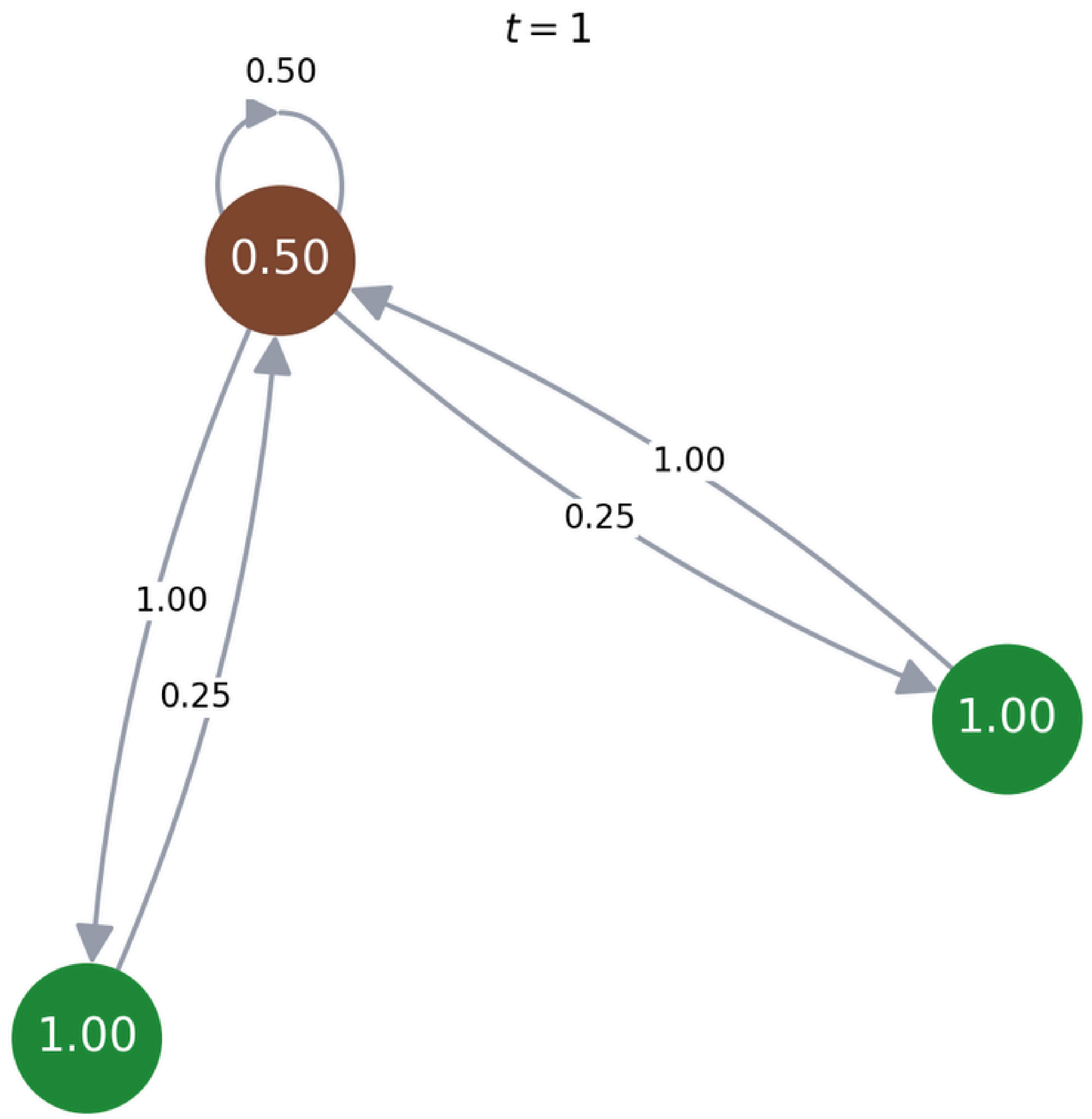
Consider the graph on the right. The weight matrix is:

$$W = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Take initial beliefs to be $x_1^0 = 1$, $x_2^0 = 0$, and $x_3^0 = 0$.

Beliefs at time $t = 1$ are:

$$\begin{aligned} \boldsymbol{x}^1 &= W \cdot \boldsymbol{x}^0 \\ &= \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.5 \cdot 1 + 0.25 \cdot 0 + 0.25 \cdot 0 \\ 1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 \\ 1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.5 \\ 1 \\ 1 \end{bmatrix}. \end{aligned}$$



THE LINEAR ALGEBRA OF CONSENSUS

Consider the graph on the right. The weight matrix is:

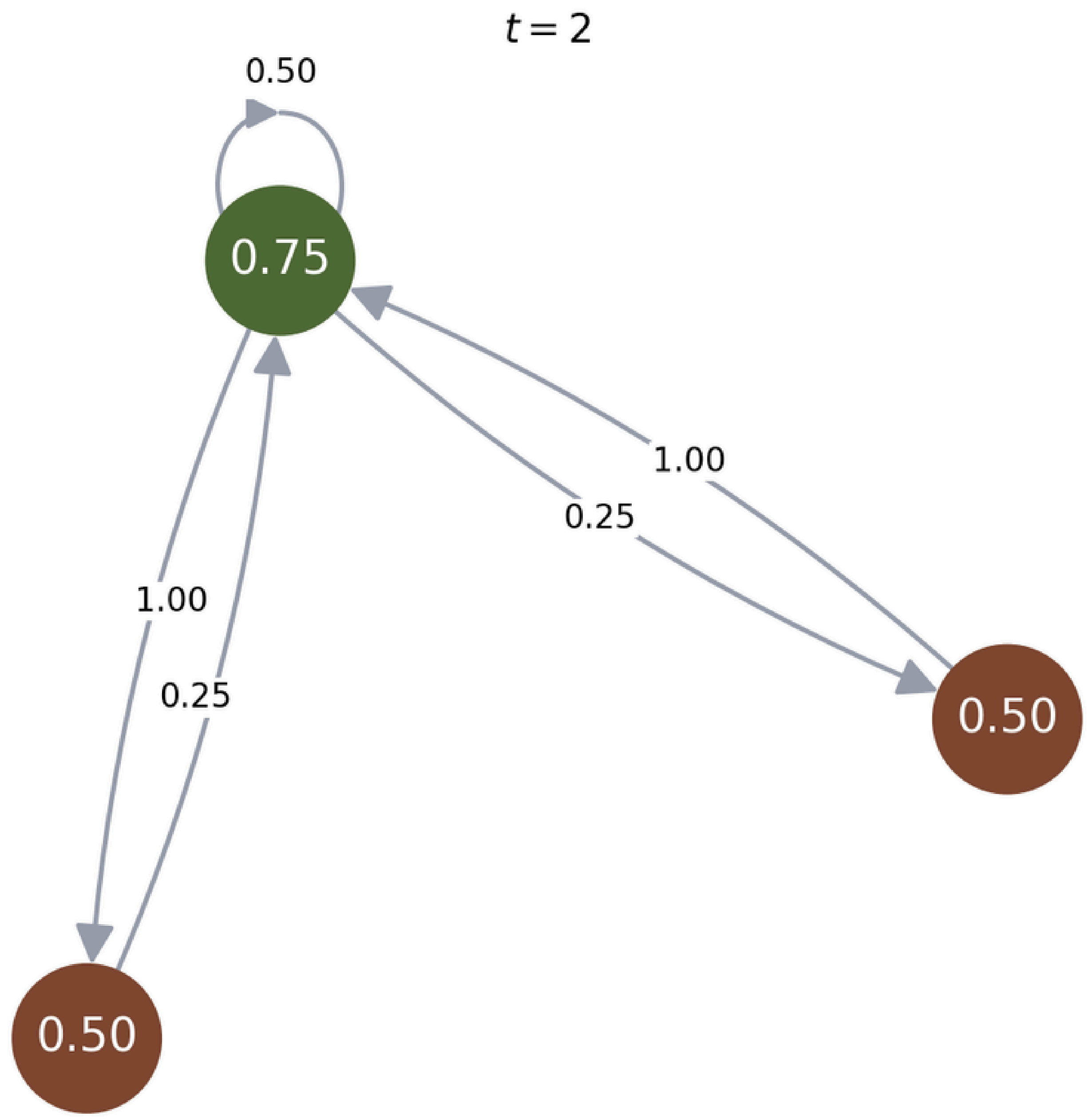
$$W = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Take initial beliefs to be $x_1^0 = 1$, $x_2^0 = 0$, and $x_3^0 = 0$.

...

Beliefs at time $t = 2$ are:

$$\begin{aligned} \boldsymbol{x}^2 &= W \cdot \boldsymbol{x}^1 \\ &= W \cdot (W \cdot \boldsymbol{x}^0) \\ &= W^2 \cdot \boldsymbol{x}^0. \\ &= \begin{bmatrix} 0.75 \\ 0.5 \\ 0.5 \end{bmatrix} \end{aligned}$$



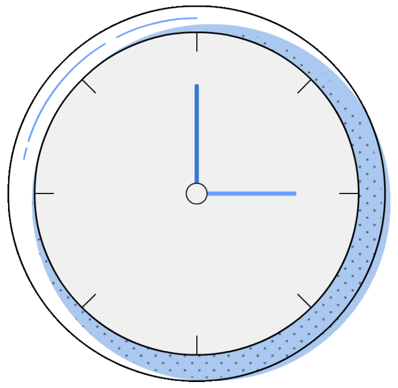
THE LINEAR ALGEBRA OF CONSENSUS

Consider the graph on the right. The weight matrix is:

$$W = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Take initial beliefs to be $x_1^0 = 1$, $x_2^0 = 0$, and $x_3^0 = 0$.

...



THE LINEAR ALGEBRA OF CONSENSUS

Consider the graph on the right. The weight matrix is:

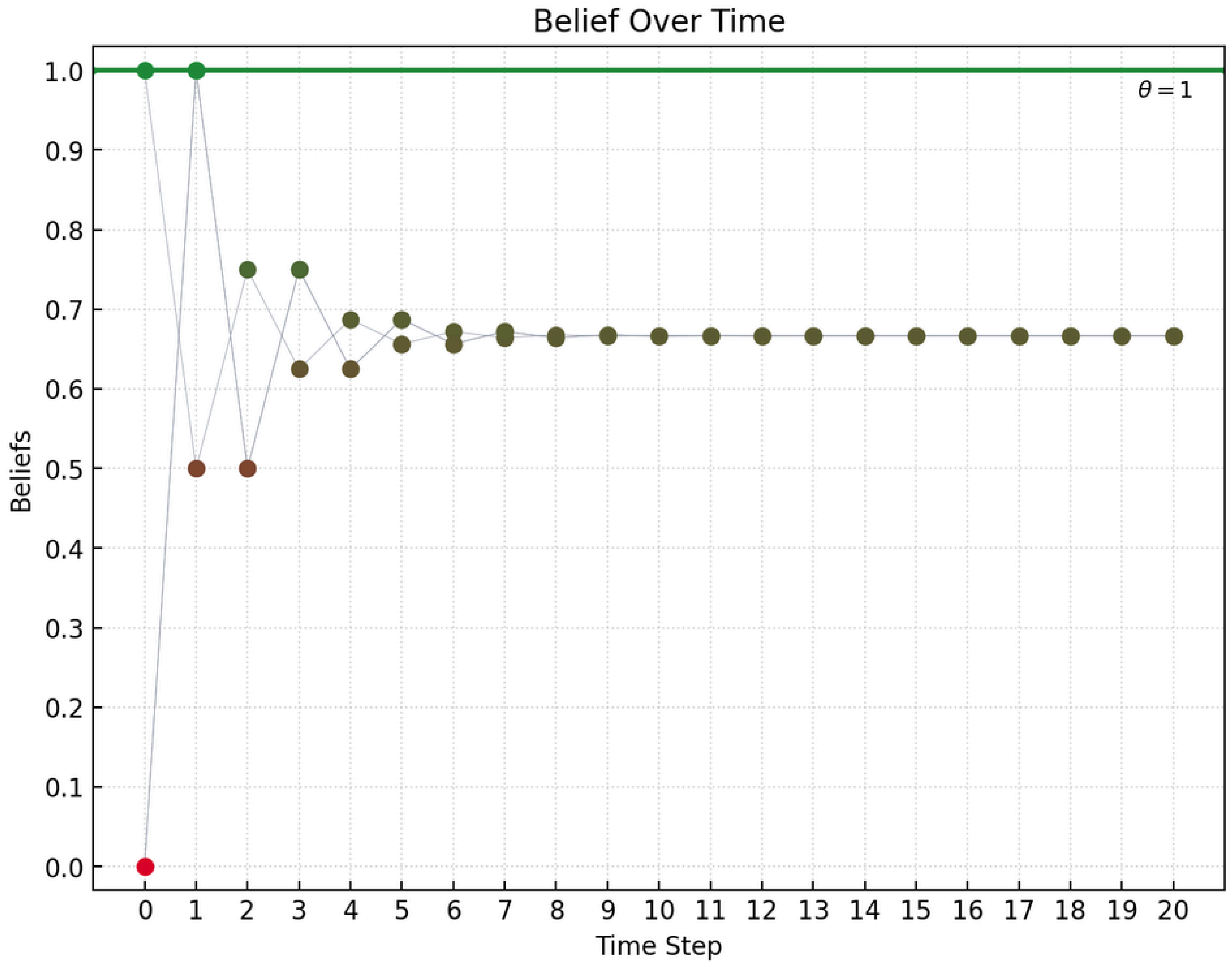
$$W = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Take initial beliefs to be $x_1^0 = 1$, $x_2^0 = 0$, and $x_3^0 = 0$.

...

In general, beliefs at time t are:

$$\boldsymbol{x}^t = W^t \cdot \boldsymbol{x}^0.$$



THE LINEAR ALGEBRA OF CONSENSUS

Consider the graph on the right. The weight matrix is:

$$W = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

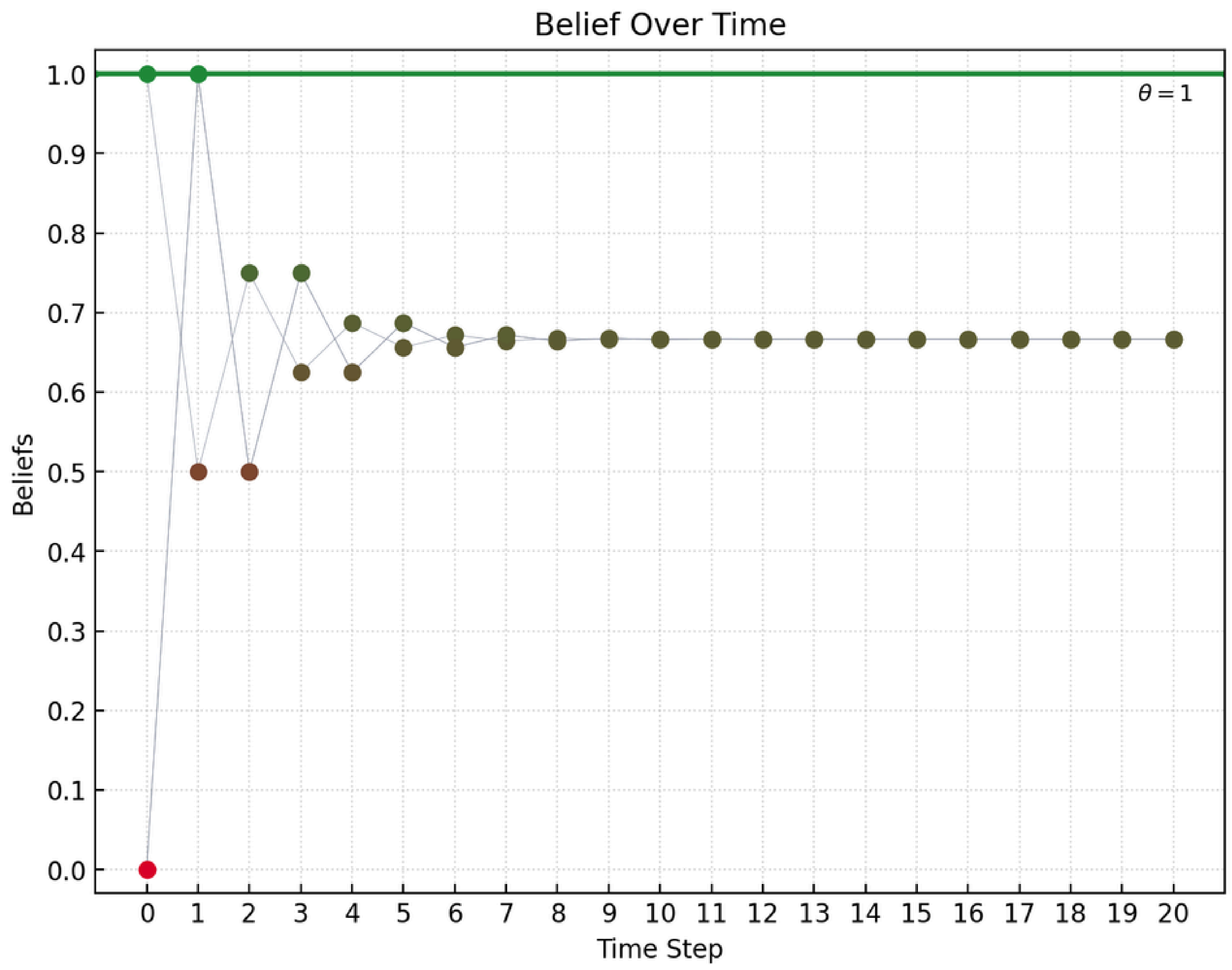
Take initial beliefs to be $x_1^0 = 1$, $x_2^0 = 0$, and $x_3^0 = 0$.

...

As t goes to infinity, the beliefs converge to a limit:*

$$\begin{aligned} \boldsymbol{x}^* &= \lim_{t\to\infty} W^t \cdot \boldsymbol{x}^0 \\ &= W^* \cdot \boldsymbol{x}^0. \\ &= \begin{bmatrix} w_1^* & w_2^* & w_3^* \\ w_1^* & w_2^* & w_3^* \\ w_1^* & w_2^* & w_3^* \end{bmatrix} \cdot \begin{bmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{bmatrix}. \end{aligned}$$

*Note that, since the limit belief is indenpendent of the initial beliefs, the rows of W^* have to be equal.



THE LINEAR ALGEBRA OF CONSENSUS

Consider the graph on the right. The weight matrix is:

$$W = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

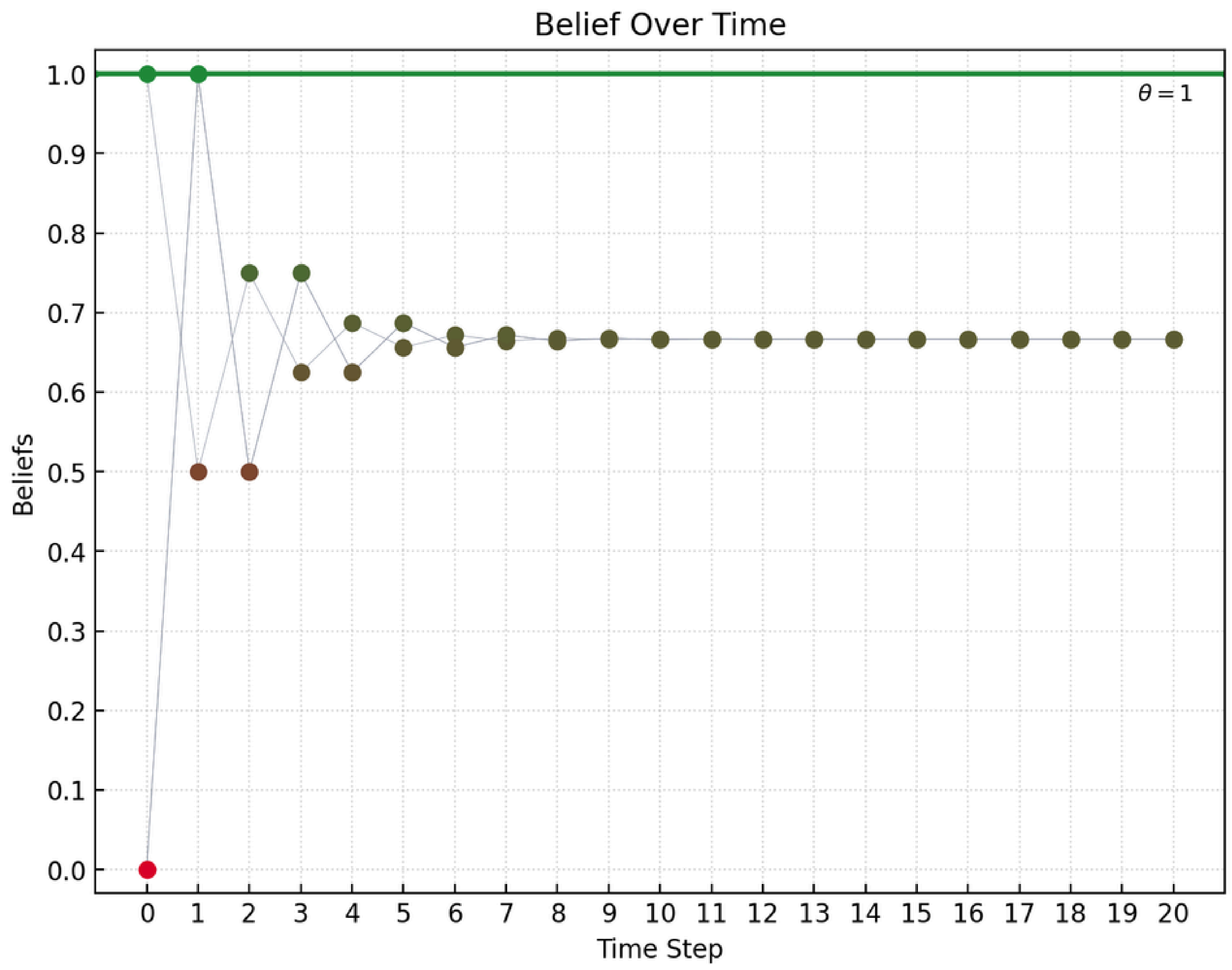
Take initial beliefs to be $x_1^0 = 1$, $x_2^0 = 0$, and $x_3^0 = 0$.

...

So the limit belief is:

$$\begin{aligned} \tilde{x} &= w_1^* x_1^0 + w_2^* x_2^0 + w_3^* x_3^0 \\ &= \begin{bmatrix} w_1^* & w_2^* & w_3^* \end{bmatrix} \cdot \begin{bmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{bmatrix} \\ &= \boldsymbol{w} \cdot \boldsymbol{x}^0. \end{aligned}$$

Note, again, that this holds for any initial beliefs.



So what are these mysterious *w*'s?

So what are these mysterious *w's*? Here's where it gets cool.

THE LINEAR ALGEBRA OF CONSENSUS

We got that the consensus belief is:

$$\tilde{x} = \boldsymbol{w} \cdot \boldsymbol{x}^0.$$

THE LINEAR ALGEBRA OF CONSENSUS

We got that the consensus belief is:

$$\tilde{x} = \boldsymbol{w} \cdot \boldsymbol{x}^0.$$

But note that we would get the same consensus belief even if we started at $\boldsymbol{x}^1$!

THE LINEAR ALGEBRA OF CONSENSUS

We got that the consensus belief is:

$$\tilde{x} = \boldsymbol{w} \cdot \boldsymbol{x}^0.$$

But note that we would get the same consensus belief even if we started at $\boldsymbol{x}^1$!

So:

$$\begin{aligned} \boldsymbol{w} \cdot \boldsymbol{x}^0 &= \boldsymbol{w} \cdot \boldsymbol{x}^1 \\ &= \boldsymbol{w} \cdot \big(W \cdot \boldsymbol{x}^0\big) \\ &= (\boldsymbol{w} \cdot W) \cdot \boldsymbol{x}^0. \end{aligned}$$

THE LINEAR ALGEBRA OF CONSENSUS

We got that the consensus belief is:

$$\tilde{x} = \boldsymbol{w} \cdot \boldsymbol{x}^0.$$

But note that we would get the same consensus belief even if we started at $\boldsymbol{x}^1$!

So:

$$\begin{aligned}
\boldsymbol{w} \cdot \boldsymbol{x}^0 &= \boldsymbol{w} \cdot \boldsymbol{x}^1 \\
&= \boldsymbol{w} \cdot \left(W \cdot \boldsymbol{x}^0\right) \\
&= (\boldsymbol{w} \cdot W) \cdot \boldsymbol{x}^0.
\end{aligned}$$

Simplifying and rearranging gives us:

$$\boldsymbol{w} \cdot W = \boldsymbol{w}.$$

THE LINEAR ALGEBRA OF CONSENSUS

We got that the consensus belief is:

$$\tilde{x} = \boldsymbol{w} \cdot \boldsymbol{x}^0.$$

But note that we would get the same consensus belief even if we started at $\boldsymbol{x}^1$!

So:

$$\begin{aligned} \boldsymbol{w} \cdot \boldsymbol{x}^0 &= \boldsymbol{w} \cdot \boldsymbol{x}^1 \\ &= \boldsymbol{w} \cdot \left(W \cdot \boldsymbol{x}^0\right) \\ &= (\boldsymbol{w} \cdot W) \cdot \boldsymbol{x}^0. \end{aligned}$$

Simplifying and rearranging gives us:

$$\boldsymbol{w} \cdot W = \boldsymbol{w}.$$

This means that $\boldsymbol{w}$ is a *left eigenvector* of W with eigenvalue 1.

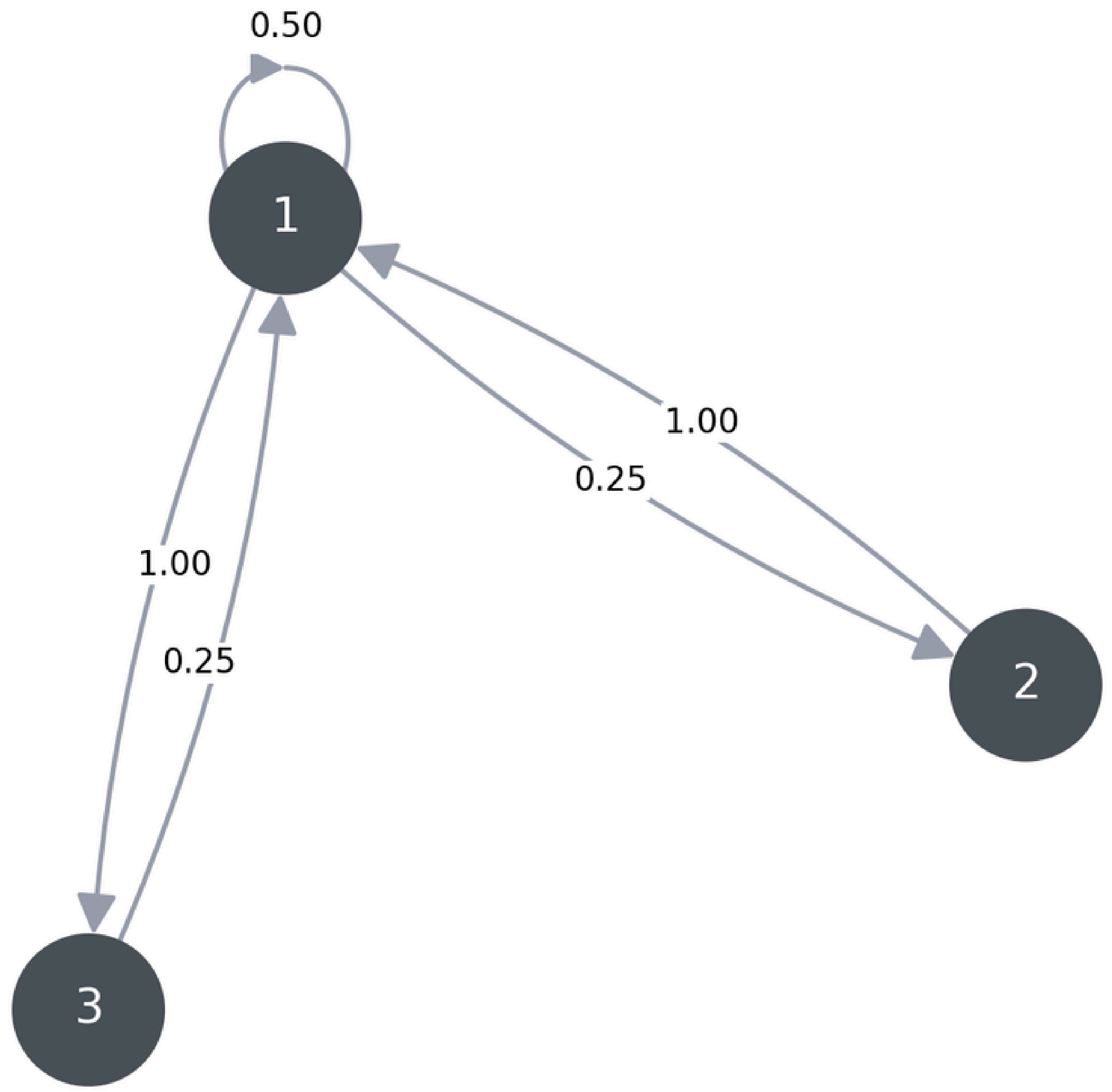
The elements of $\boldsymbol{w}$ are exactly the *eigenvector centralities* of the nodes!

The elements of $\boldsymbol{w}$ are exactly the *eigenvector centralities* of the nodes! So we just got that the consensus belief is a linear combination of the initial beliefs and eigenvector centralities.

THE LINEAR ALGEBRA OF CONSENSUS

Consider the graph on the right. The weight matrix is:

$$W = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

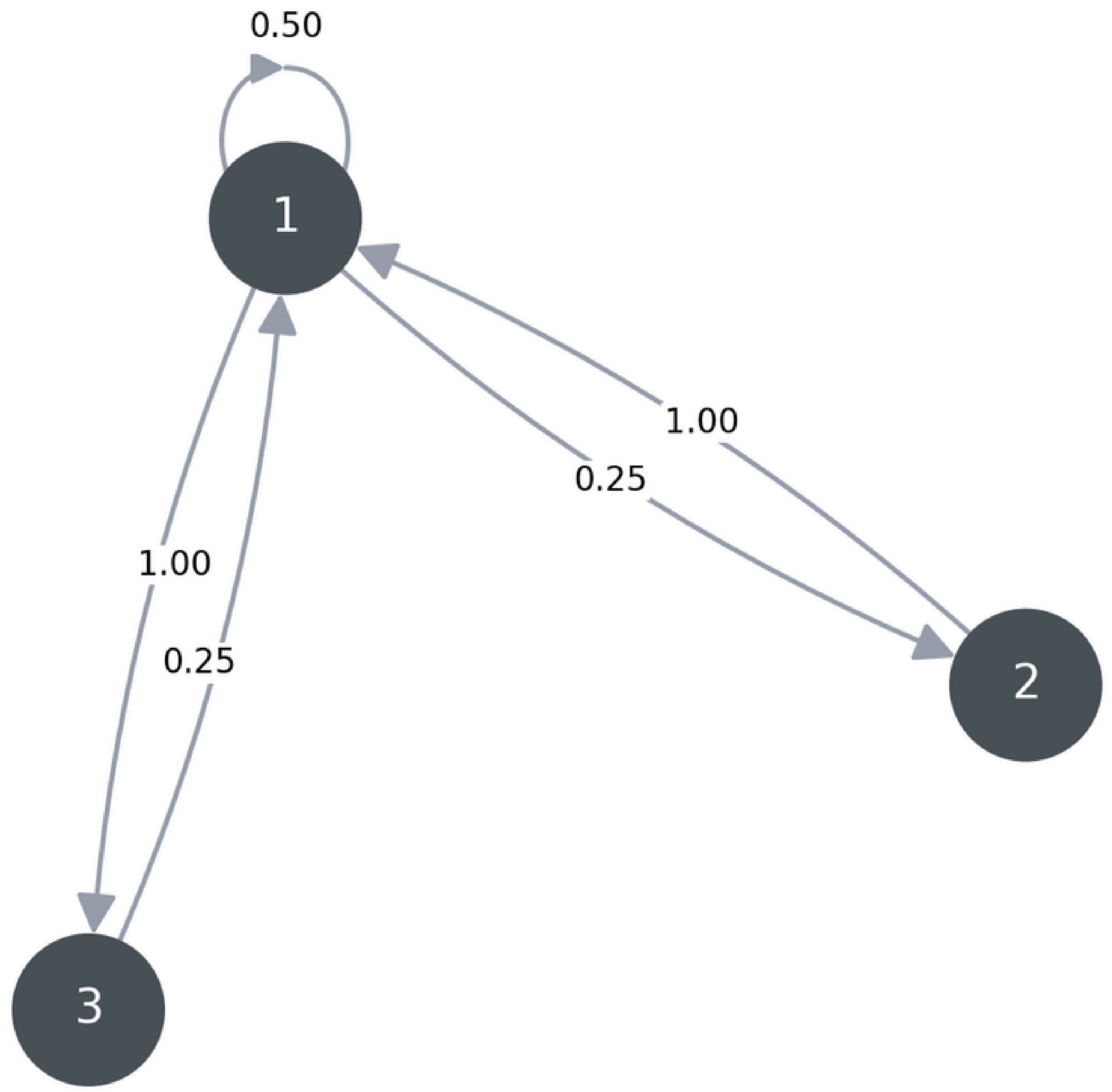


THE LINEAR ALGEBRA OF CONSENSUS

Consider the graph on the right. The weight matrix is:

$$W = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The eigenvector centralities of the nodes are $c_1 = {}^2\!/_3$, $c_2 = {}^1\!/_6$, and $c_3 = {}^1\!/_6$.



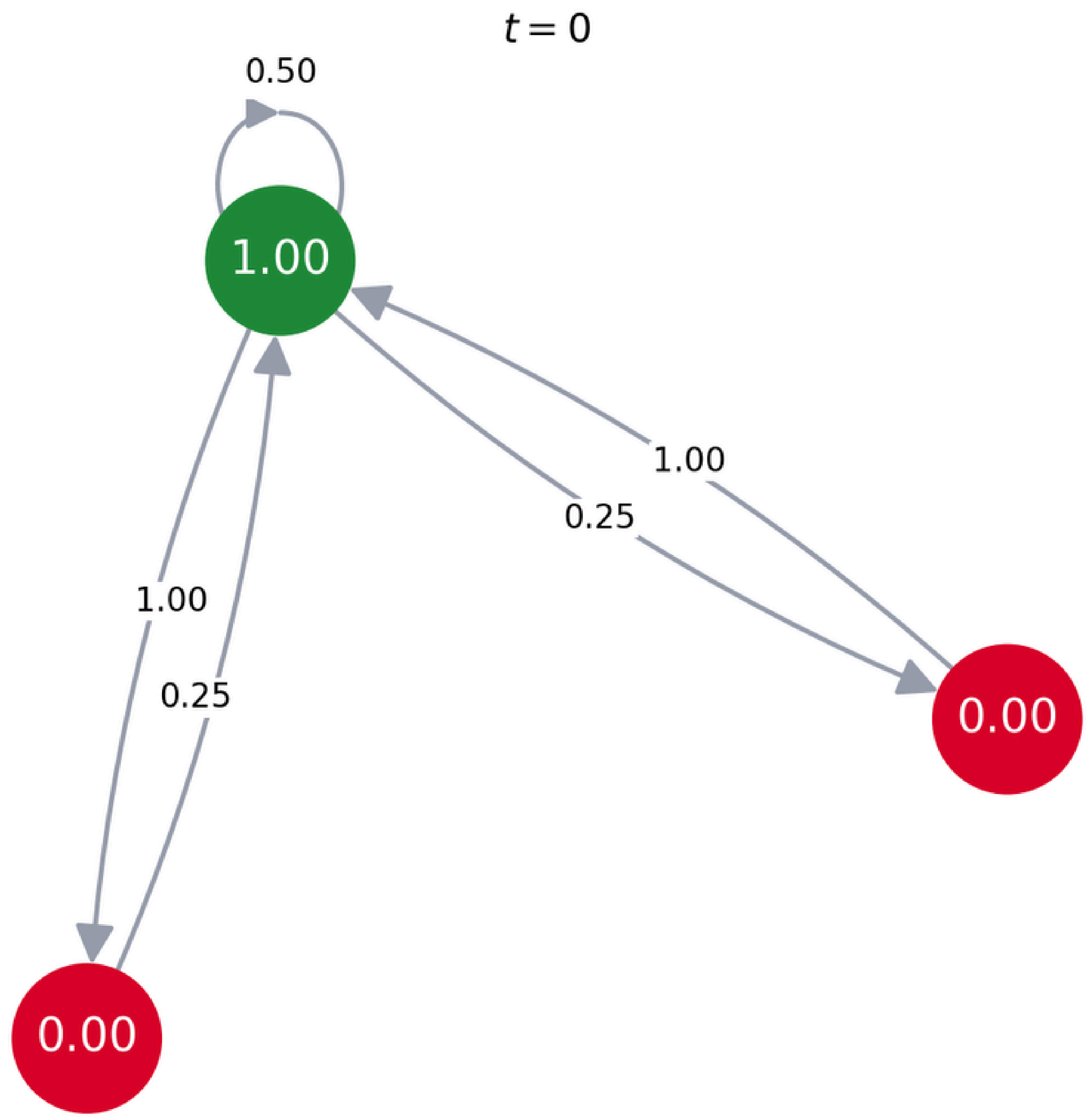
THE LINEAR ALGEBRA OF CONSENSUS

Consider the graph on the right. The weight matrix is:

$$W = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The eigenvector centralities of the nodes are $c_1 = {}^2/_3$, $c_2 = {}^1/_6$, and $c_3 = {}^1/_6$.

Take initial beliefs to be $x_1^0 = 1$, $x_2^0 = 0$, and $x_3^0 = 0$.



THE LINEAR ALGEBRA OF CONSENSUS

Consider the graph on the right. The weight matrix is:

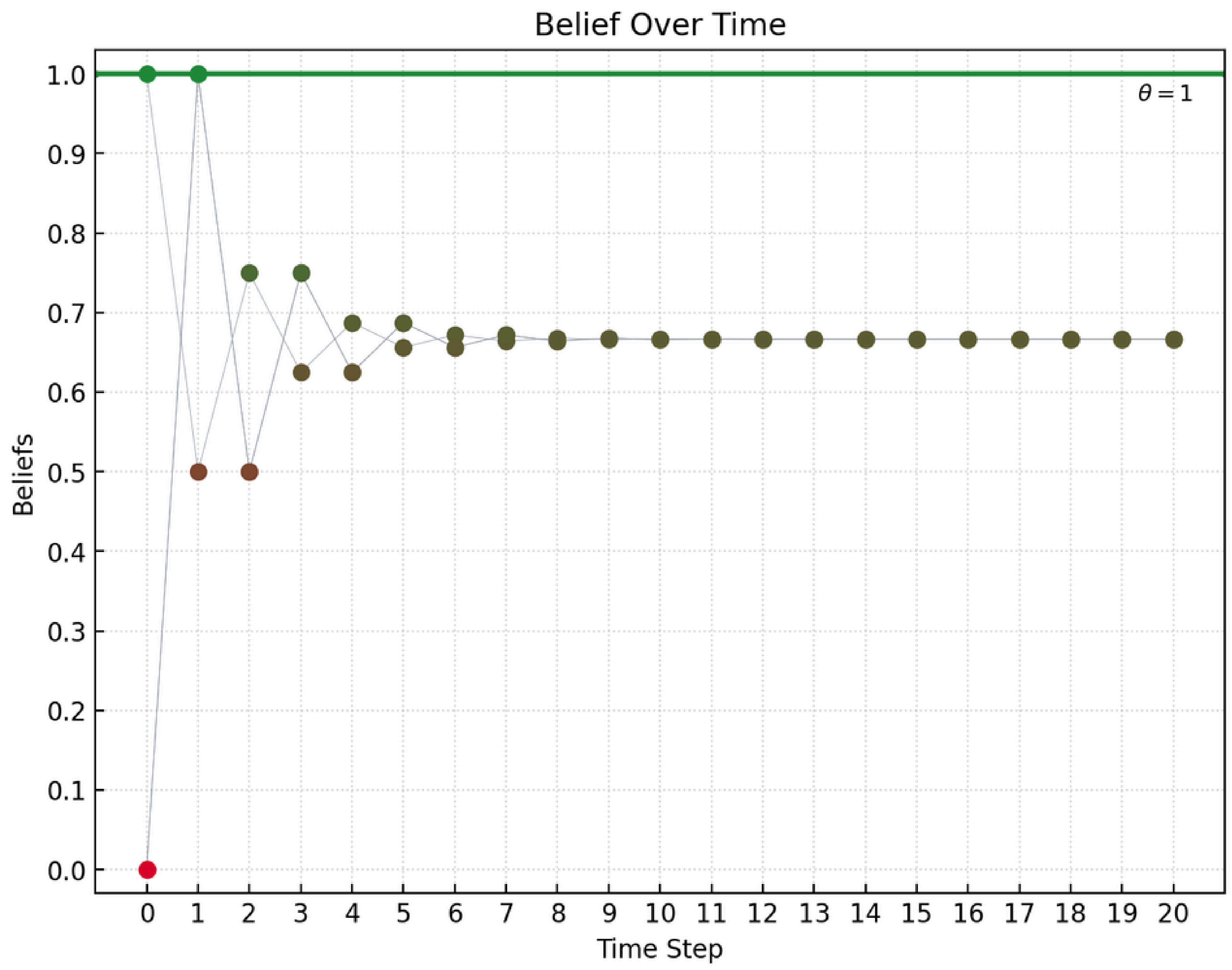
$$W = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The eigenvector centralities of the nodes are $c_1 = {}^2/_3$, $c_2 = {}^1/_6$, and $c_3 = {}^1/_6$.

Take initial beliefs to be $x_1^0 = 1$, $x_2^0 = 0$, and $x_3^0 = 0$.

If we compute the limits and do the maths, we get the consensus belief as:

$$\tilde{x} = \frac{2}{3}.$$



THE LINEAR ALGEBRA OF CONSENSUS

Consider the graph on the right. The weight matrix is:

$$W = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The eigenvector centralities of the nodes are $c_1 = {}^2/_3$, $c_2 = {}^1/_6$, and $c_3 = {}^1/_6$.

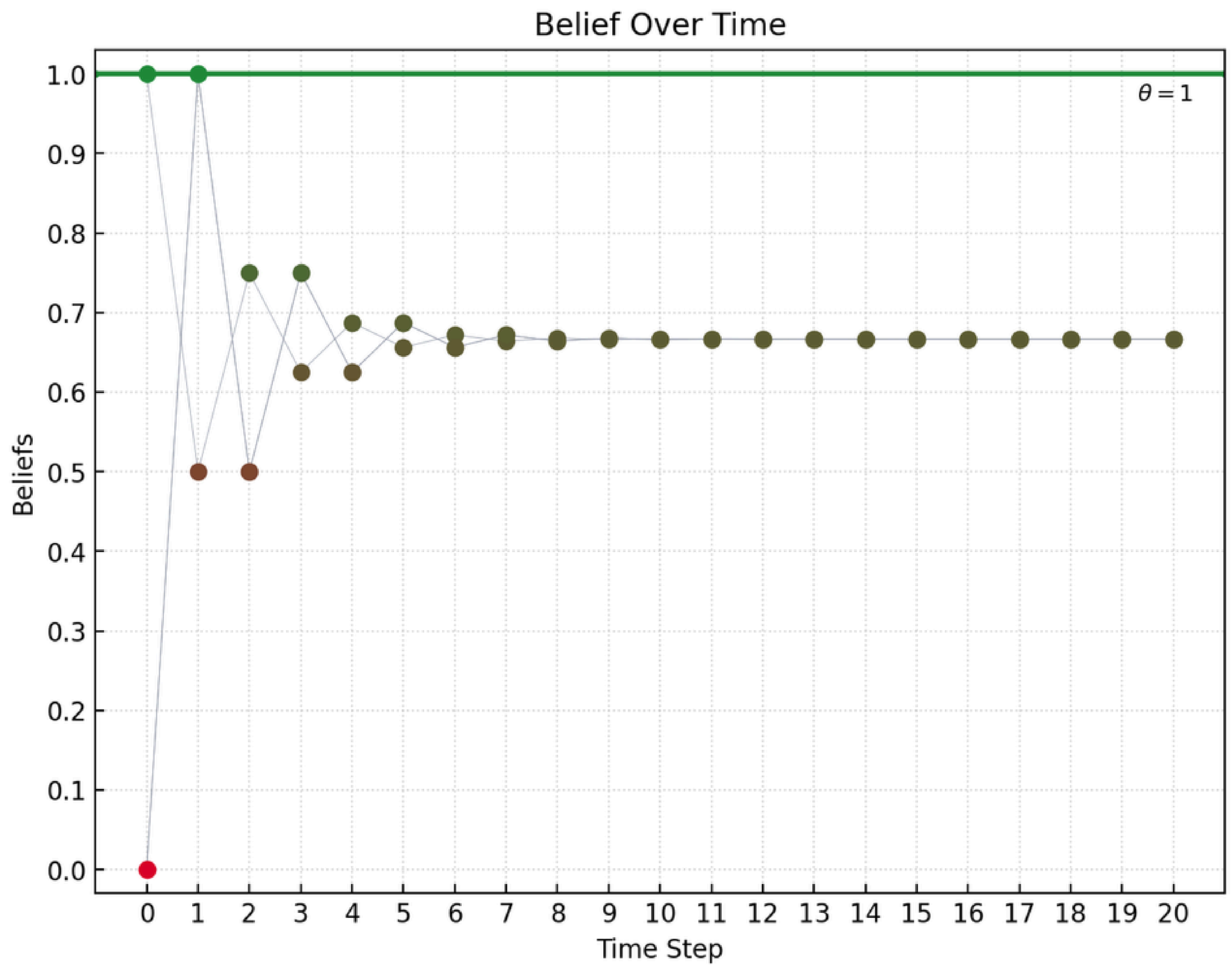
Take initial beliefs to be $x_1^0 = 1$, $x_2^0 = 0$, and $x_3^0 = 0$.

If we compute the limits and do the maths, we get the consensus belief as:

$$\tilde{x} = \frac{2}{3}.$$

Note that we get the same from the eigenvector centralities and initial beliefs:

$$\begin{aligned} c_1 \cdot x_1^0 + c_2 \cdot x_2^0 + c_3 \cdot x_3^0 &= \frac{2}{3} \cdot 1 + \frac{1}{6} \cdot 0 + \frac{1}{6} \cdot 0 \\ &= \frac{2}{3}. \end{aligned}$$



THEOREM (GOLUB & JACKSON, 2010)
Assume a sequence $G_1, G_2, \ldots,$ of strongly connected and aperiodic networks of increasing size, and initial beliefs drawn from a distribution with mean μ (the true state) and finite variance above a threshold $\delta > 0$.

The sequence of networks is *wise* if and only if the eigenvector centrality of every agent approaches 0 asymptotically, as n goes to infinity.

Golub, B., & Jackson, M. O. (2010). Naïve Learning in Social Networks and the Wisdom of Crowds. *American Economic Journal: Microeconomics*, 2(1), 112–149.

BENJAMIN GOLUB

For a network to be wise, there can’t be a node that, in the long run, retains positive influence.

BENJAMIN GOLUB

For a network to be wise, there can't be a node that, in the long run, retains positive influence.

MATTHEW O. JACKSON

As the network grows and grows, the influence of every node should go to 0.

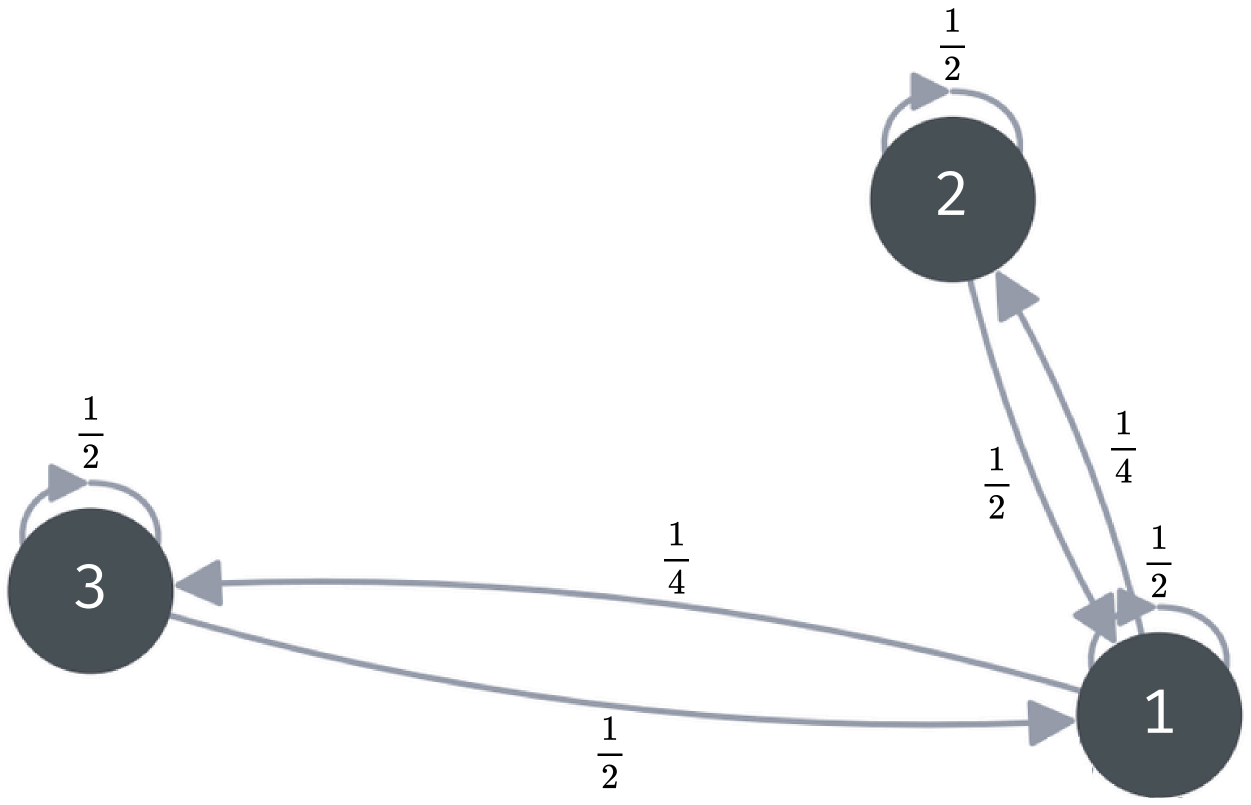
Golub, B., & Jackson, M. O. (2010). Naïve Learning in Social Networks and the Wisdom of Crowds. *American Economic Journal: Microeconomics*, 2(1), 112–149.

What happens when things go wrong?

BAD NETWORKS

Take a network with a node (1), the ‘influencer’, who always gives itself a weight of ½.

The other nodes listen to node 1.



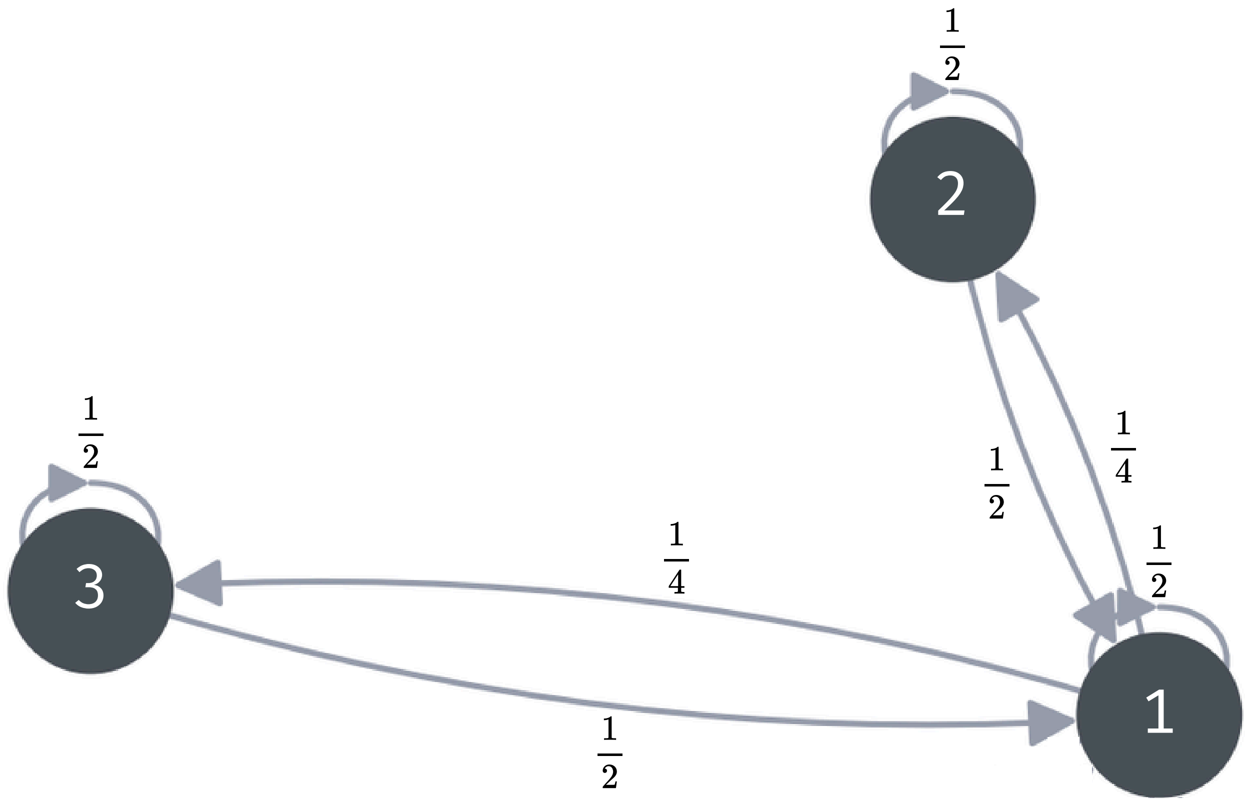
BAD NETWORKS

Take a network with a node (1), the ‘influencer’, who always gives itself a weight of ½.

The other nodes listen to node 1.

Say we start from initial beliefs:

$$x_1^0 = 1, x_2^0 = 0, x_3^0 = 0$$



BAD NETWORKS

Take a network with a node (1), the 'influencer', who always gives itself a weight of ½.

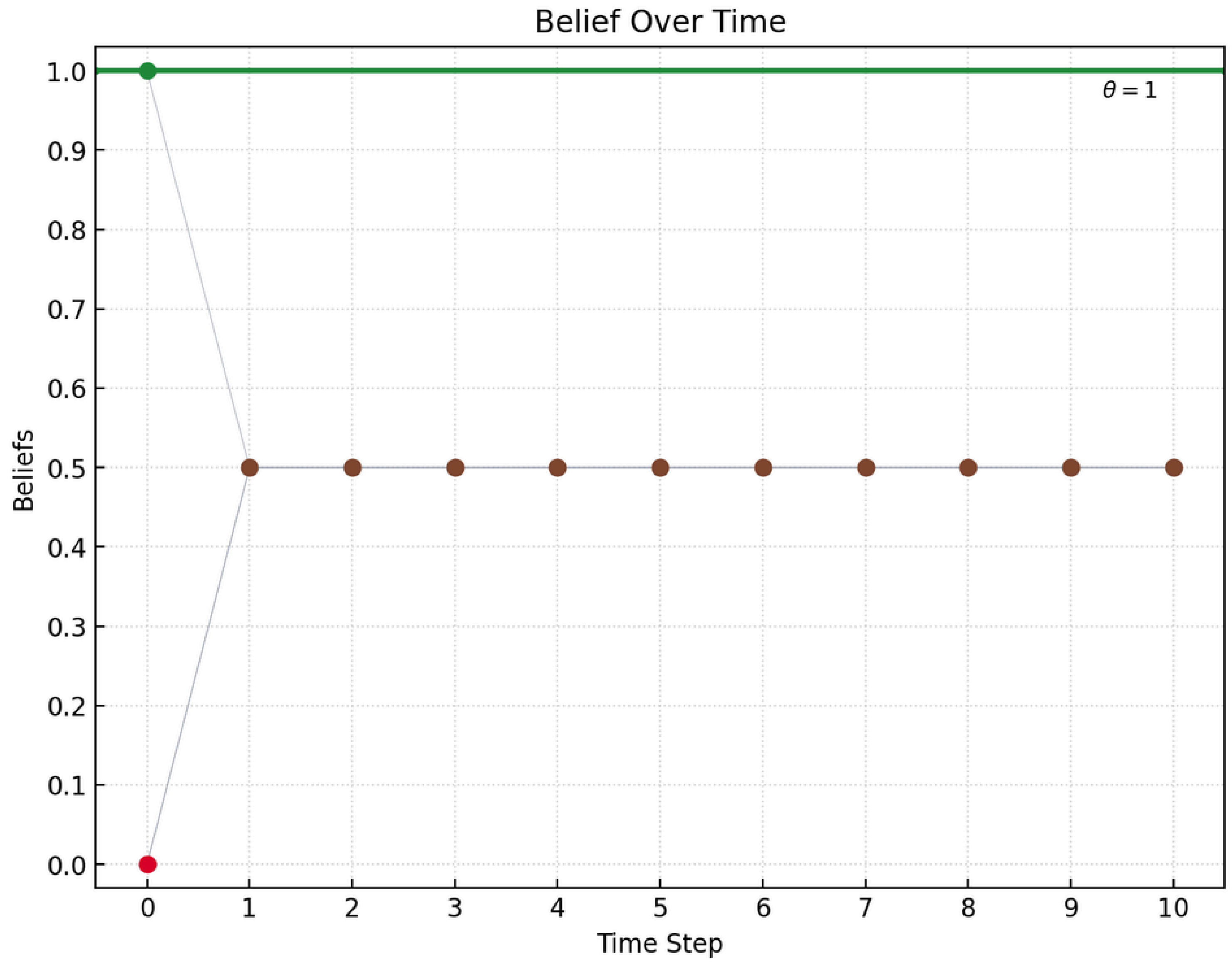
The other nodes listen to node 1.

Say we start from initial beliefs:

$$x_1^0 = 1, x_2^0 = 0, x_3^0 = 0$$

Consensus occurs at:

$$\tilde{x} = \frac{1}{2}$$



BAD NETWORKS

Take a network with a node (1), the ‘influencer’, who always gives itself a weight of ½.

The other nodes listen to node 1.

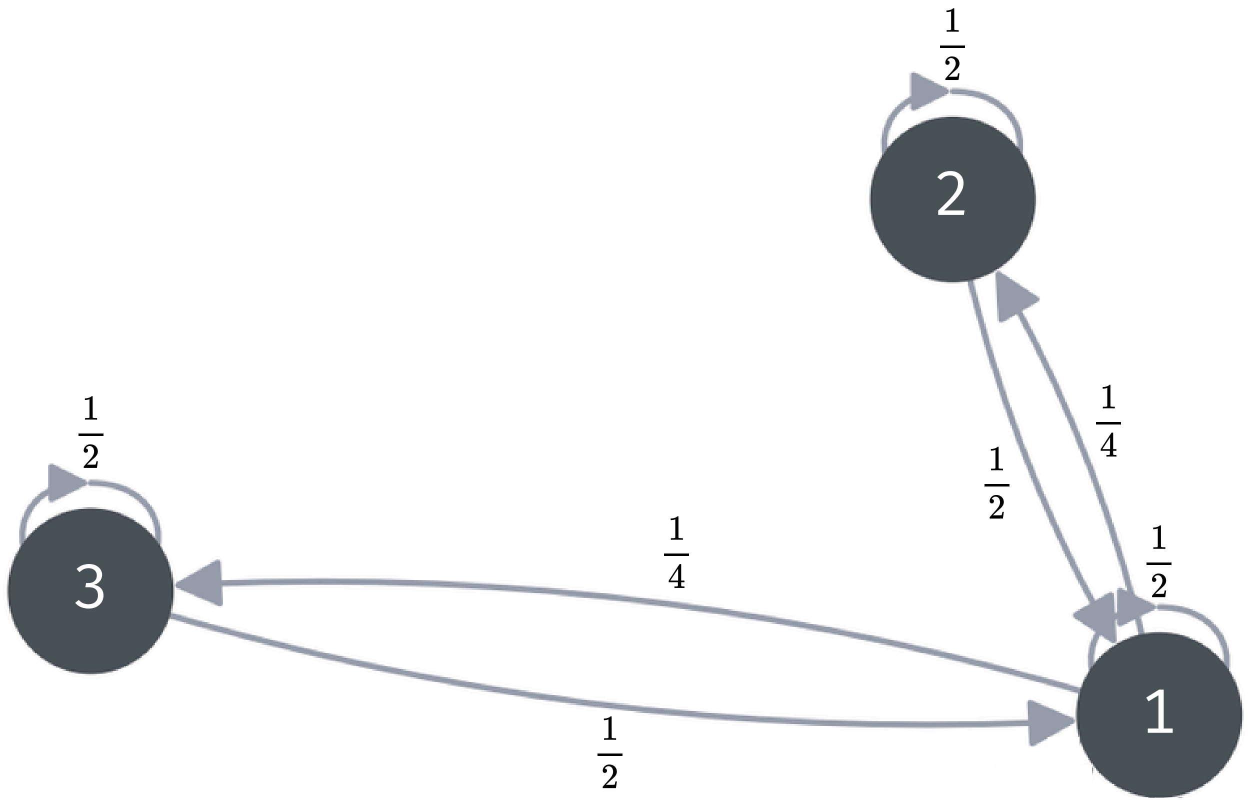
Say we start from initial beliefs:

$$x_1^0 = 1, x_2^0 = 0, x_3^0 = 0$$

Consensus occurs at:

$$\tilde{x} = \frac{1}{2}$$

Keep expanding the graph by adding nodes that listen to node 1.



BAD NETWORKS

Take a network with a node (1), the 'influencer', who always gives itself a weight of ½.

The other nodes listen to node 1.

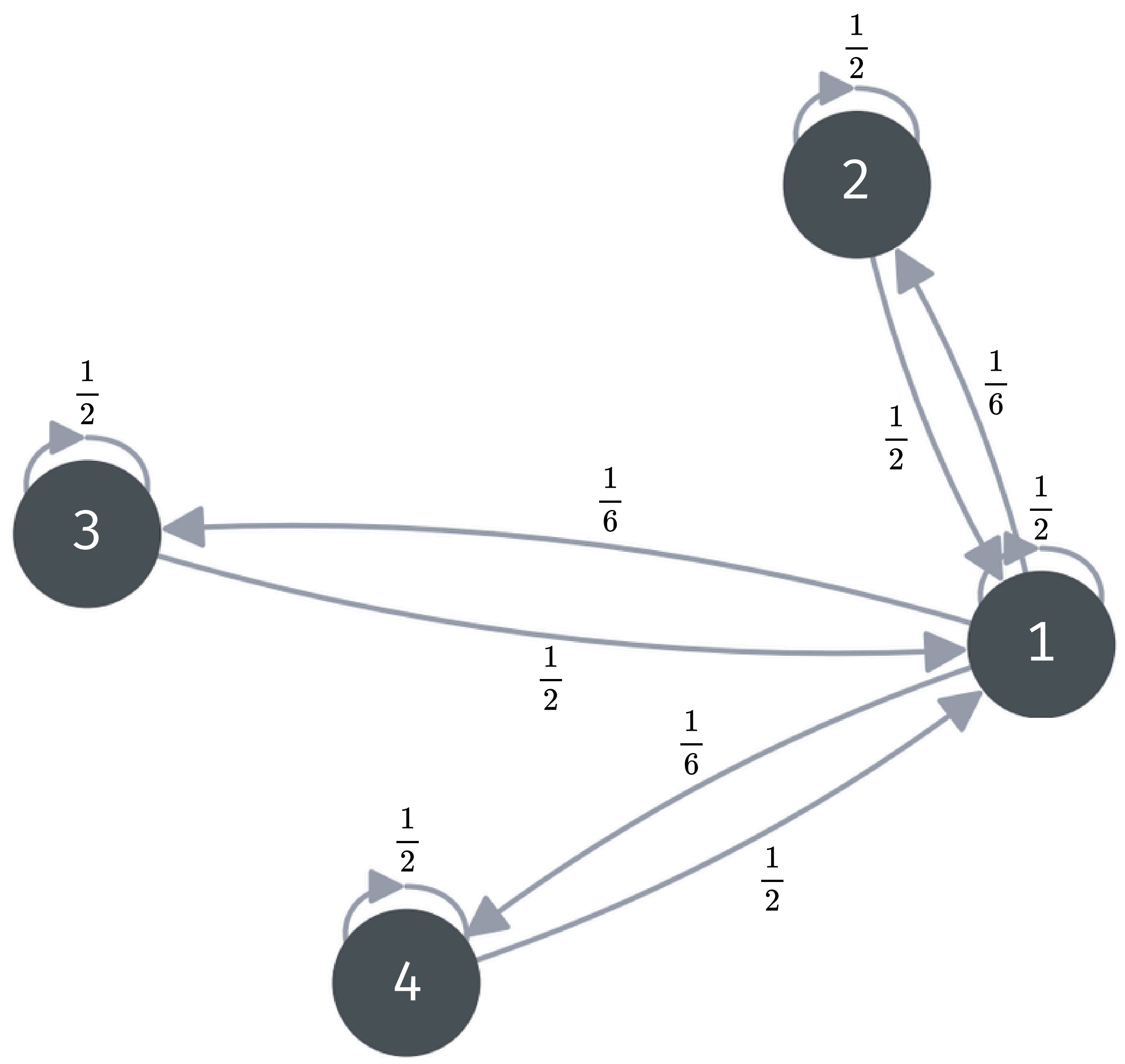
Say we start from initial beliefs:

$$x_1^0 = 1, x_2^0 = 0, x_3^0 = 0$$

Consensus occurs at:

$$\tilde{x} = \frac{1}{2}$$

Keep expanding the graph by adding nodes that listen to node 1.



BAD NETWORKS

Take a network with a node (1), the ‘influencer’, who always gives itself a weight of ½.

The other nodes listen to node 1.

Say we start from initial beliefs:

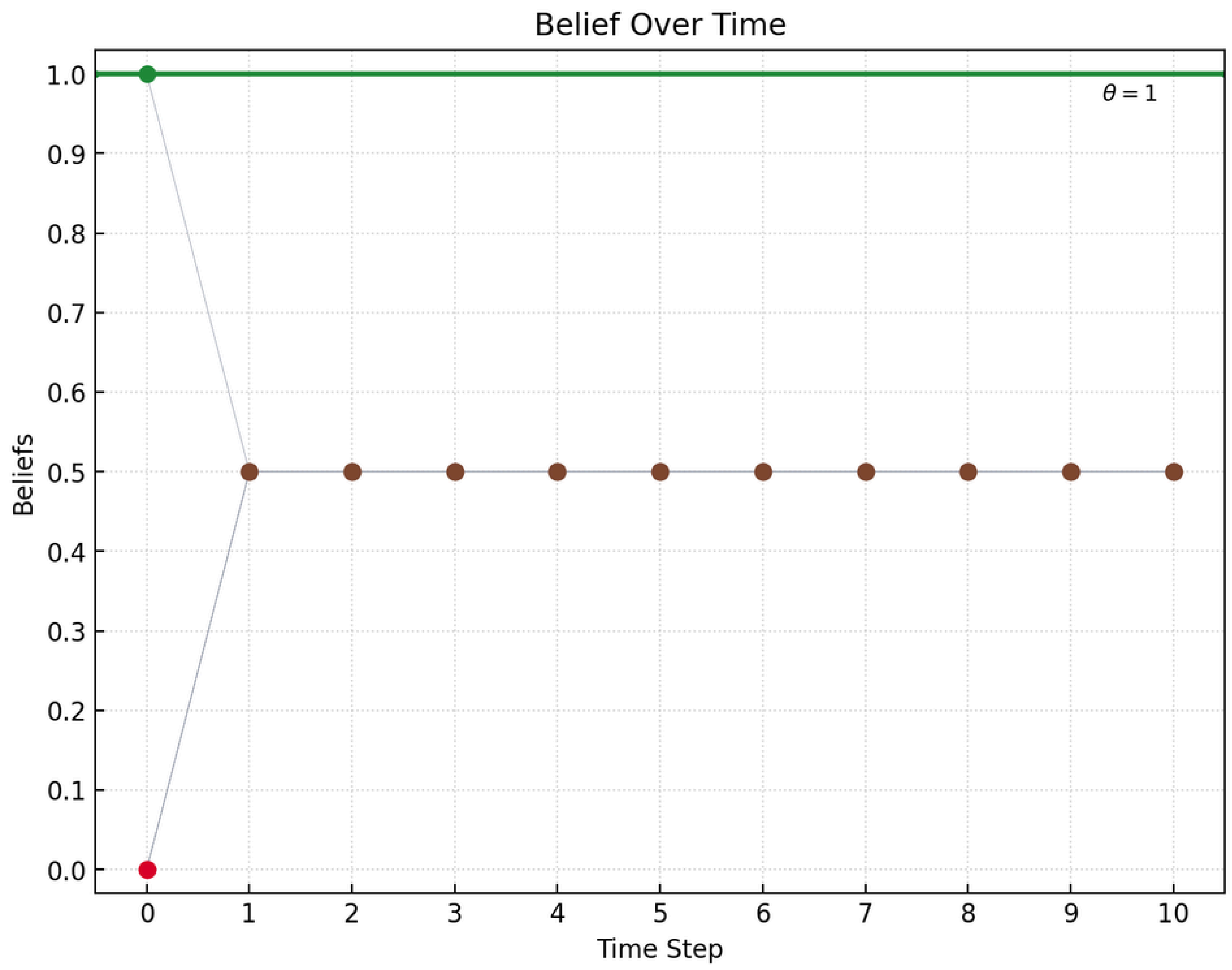
$$x_1^0 = 1, x_2^0 = 0, x_3^0 = 0$$

Consensus occurs at:

$$\tilde{x} = \frac{1}{2}$$

Keep expanding the graph by adding nodes that listen to node 1.

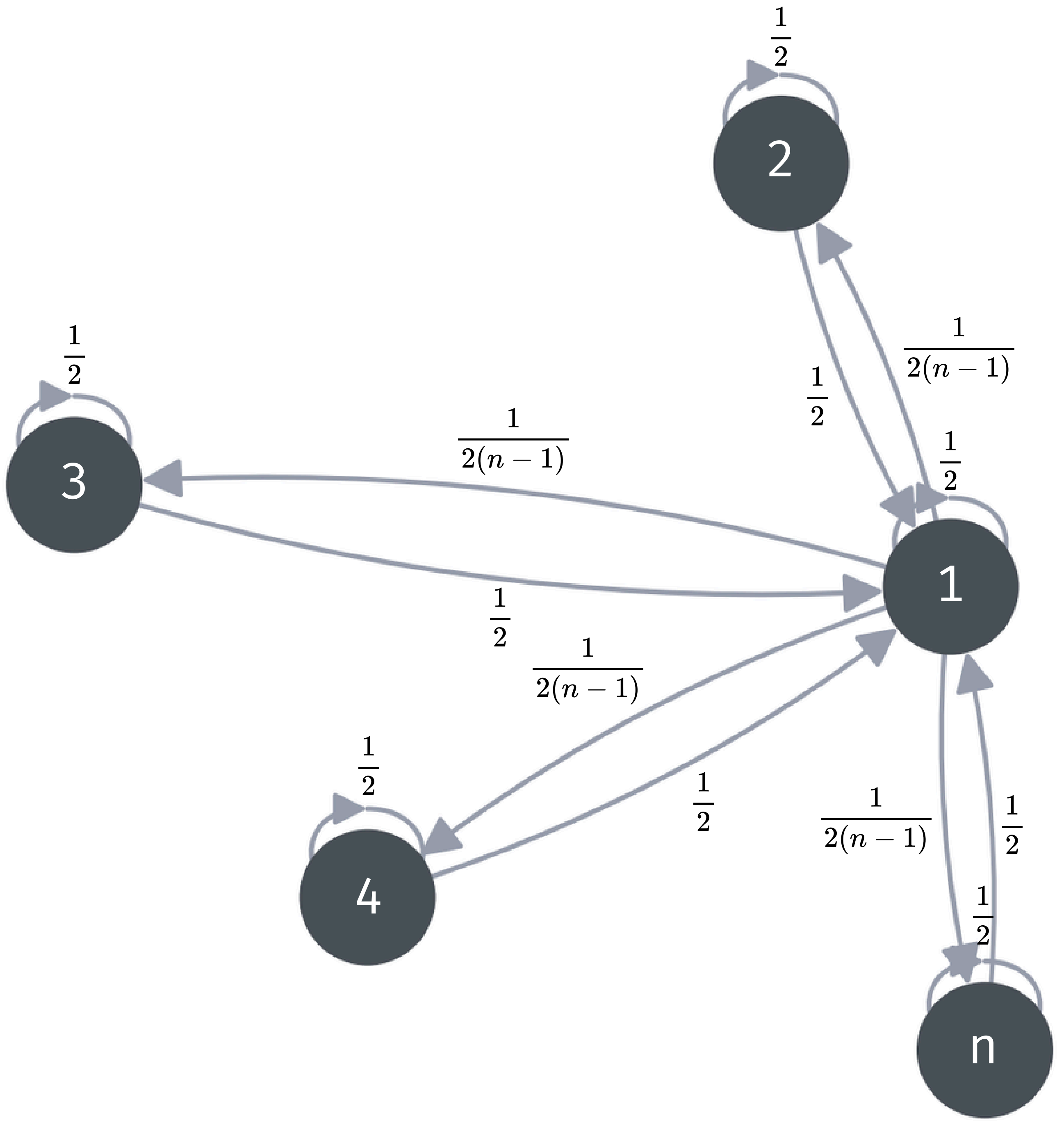
Same consensus.



What is going on?

BAD NETWORKS

The network grows by adding agents that listen to the central agent 1.



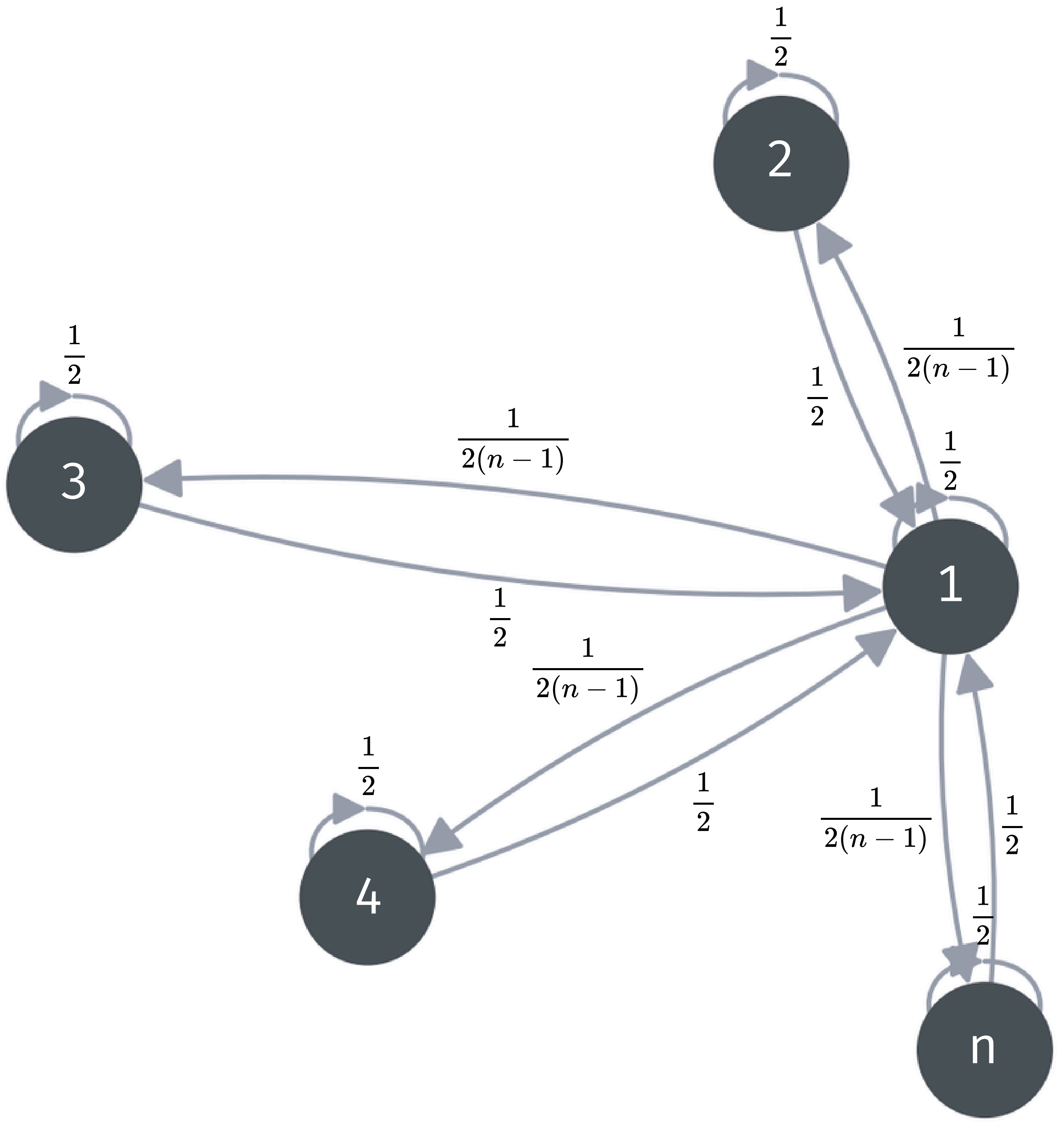
BAD NETWORKS

The network grows by adding agents that listen to the central agent 1.

The eigenvector centralities are:

$$\boldsymbol{c} = \left(\frac{1}{2}, \frac{1}{2(n-1)}, \dots, \frac{1}{2(n-1)}\right)$$

Agent 1 retains a constant share of (network) influence as n grows.



BAD NETWORKS

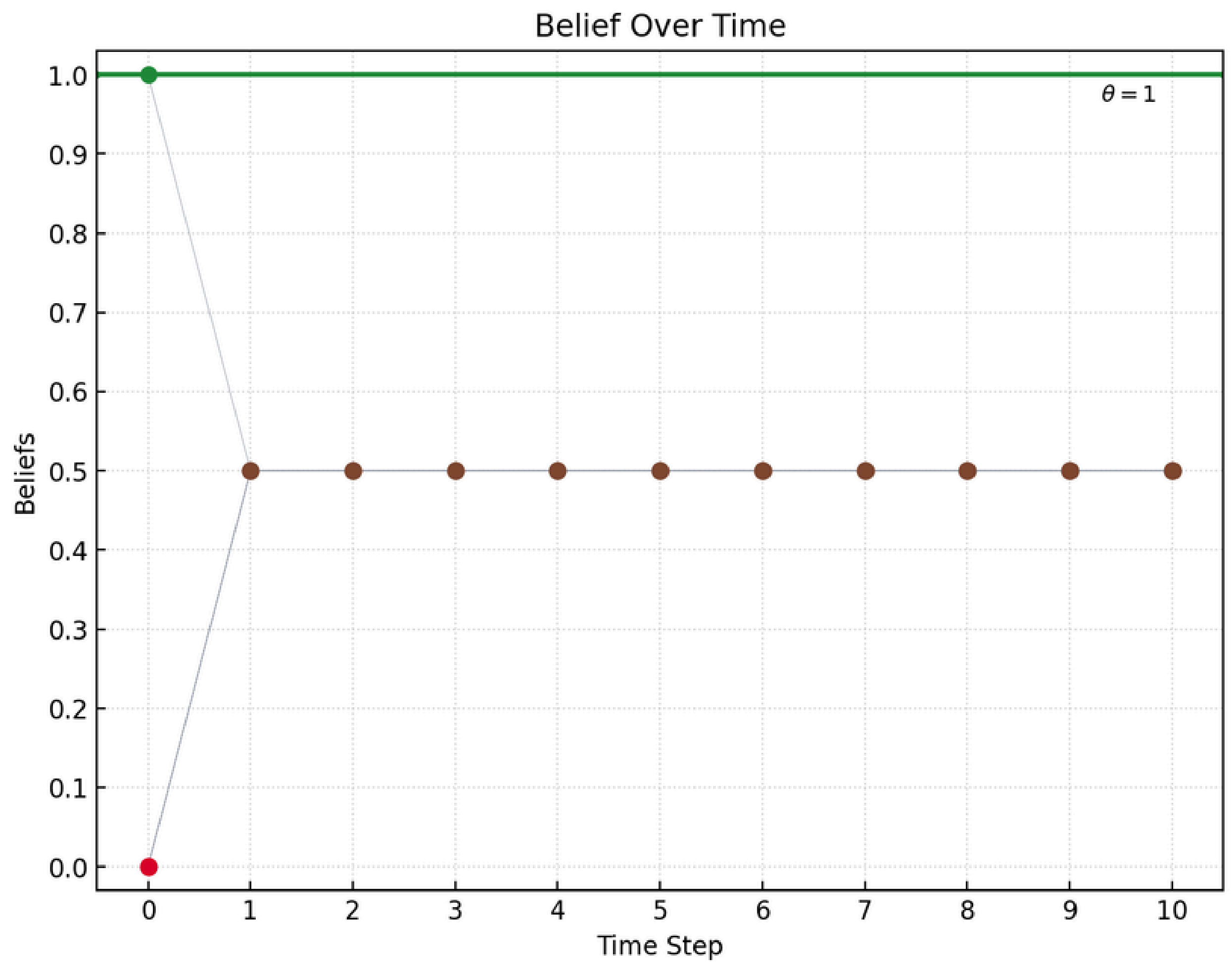
The network grows by adding agents that listen to the central agent 1.

The eigenvector centralities are:

$$\boldsymbol{c} = \left(\frac{1}{2}, \frac{1}{2(n-1)}, \ldots, \frac{1}{2(n-1)}\right)$$

Agent 1 retains a constant share of (network) influence as n grows.

And thus decides the consensus belief.



BAD NETWORKS

The network grows by adding agents that listen to the central agent 1.

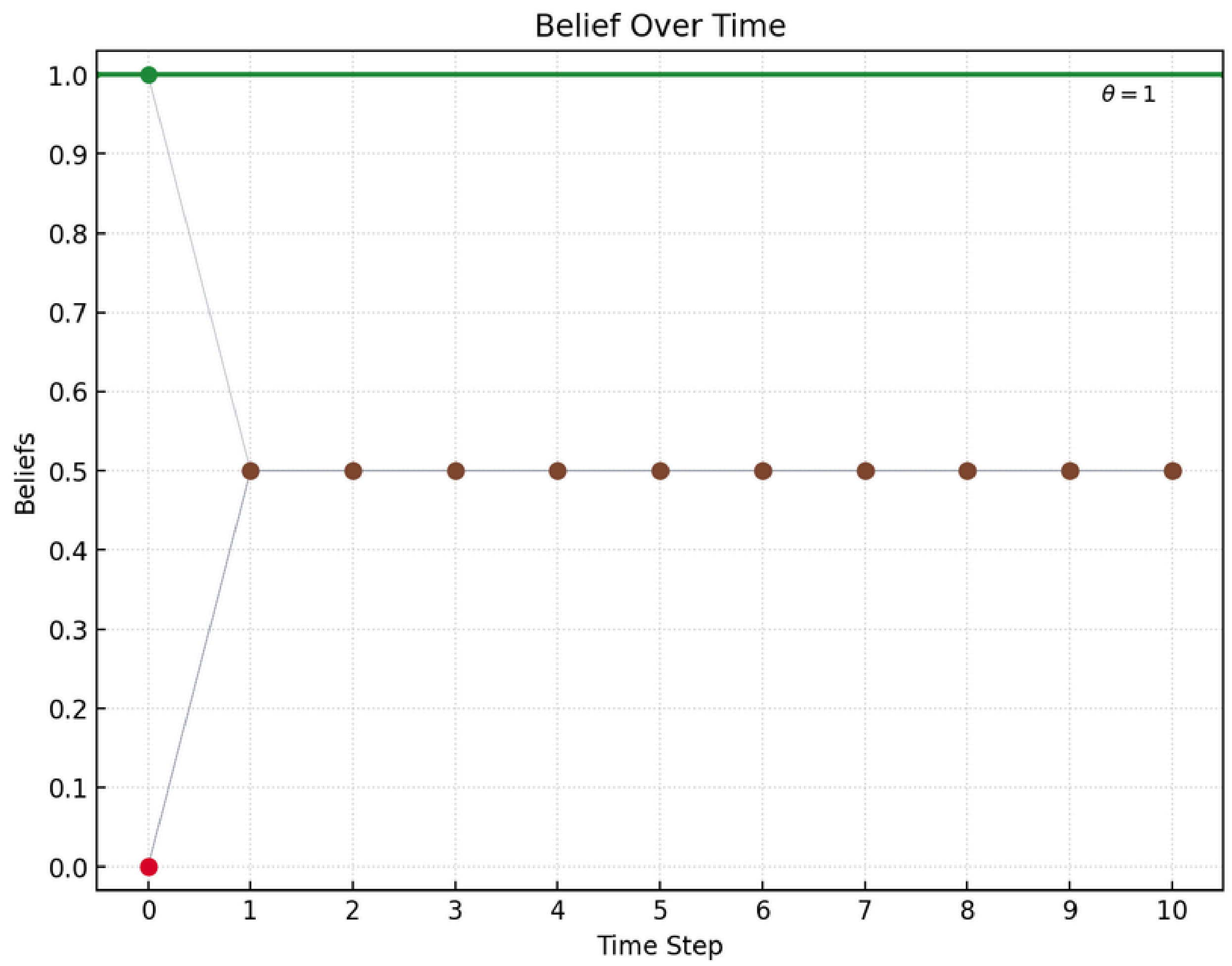
The eigenvector centralities are:

$$\boldsymbol{c} = \left(\frac{1}{2}, \frac{1}{2(n-1)}, \ldots, \frac{1}{2(n-1)}\right)$$

Agent 1 retains a constant share of (network) influence as n grows.

And thus decides the consensus belief.

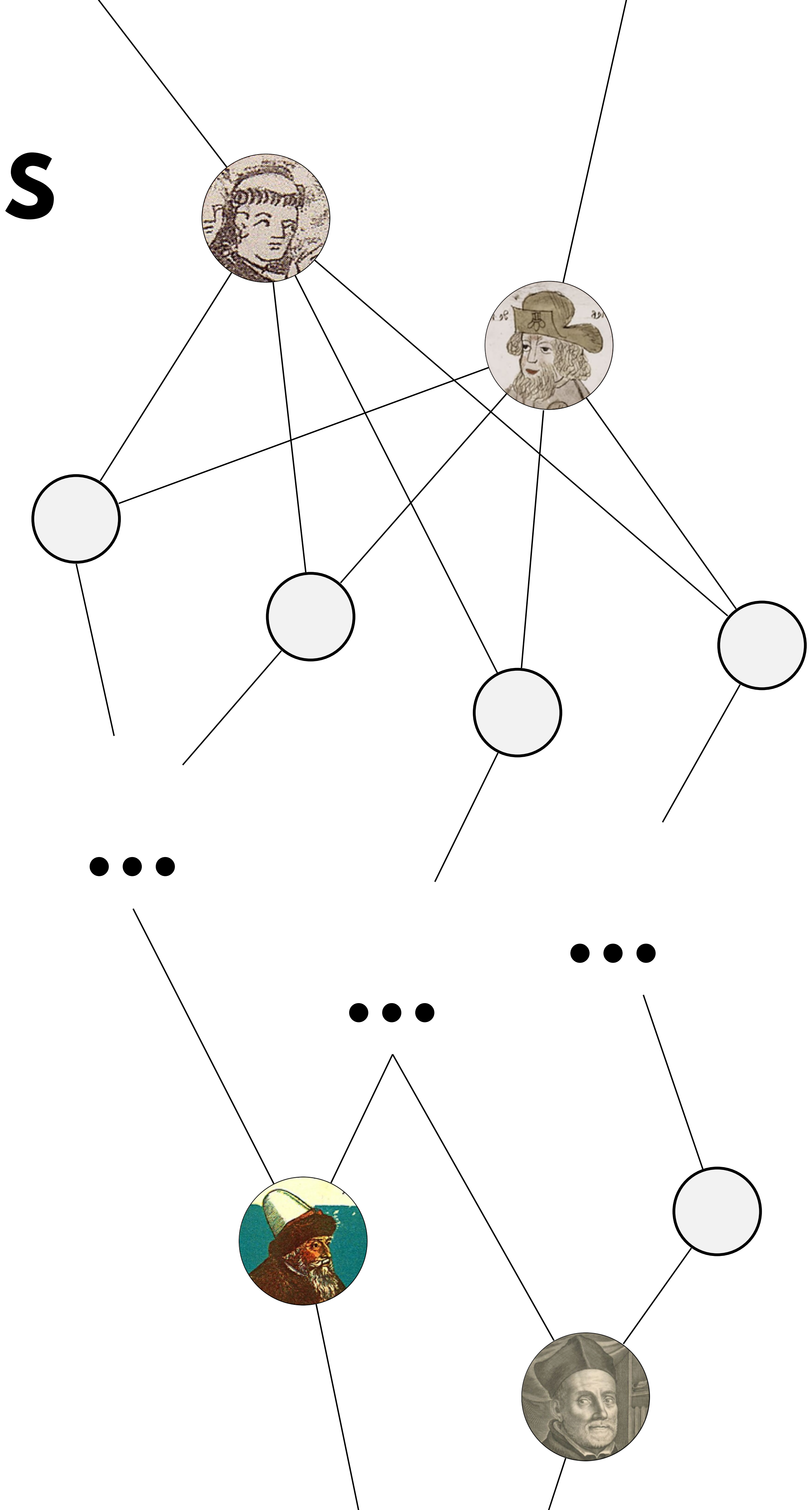
No bueno.



Influential nodes draw the collective opinion towards their own opinion, rather than the truth.

MEANWHILE, IN THE MIDDLE AGES

Maybe what happened with the vegetable lamb…



Here's a final thought.

ELON MUSK

Free speech is the bedrock of a functioning democracy.

And Twitter is the digital town square where matters vital to the future of humanity are debated.

But the shape of the social network means that some agents have an outsized influence on collective opinion.

But the shape of the social network means that some agents have an outsized influence on collective opinion. Is this still in line with democratic ideals?…