LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

NOVEMBER 25, 2024

ADVENTURES IN DEMOCRATIC DECISION MAKING

CONDORCET CONSISTENCY

Adrian Haret
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Around the same time as Borda, another French intellectual was making his voice heard...

MARIE JEAN ANTOINE NICOLAS DE CARITAT, MARQUIS OF CONDORCET

1743 – 1794

Born into an ancient family.

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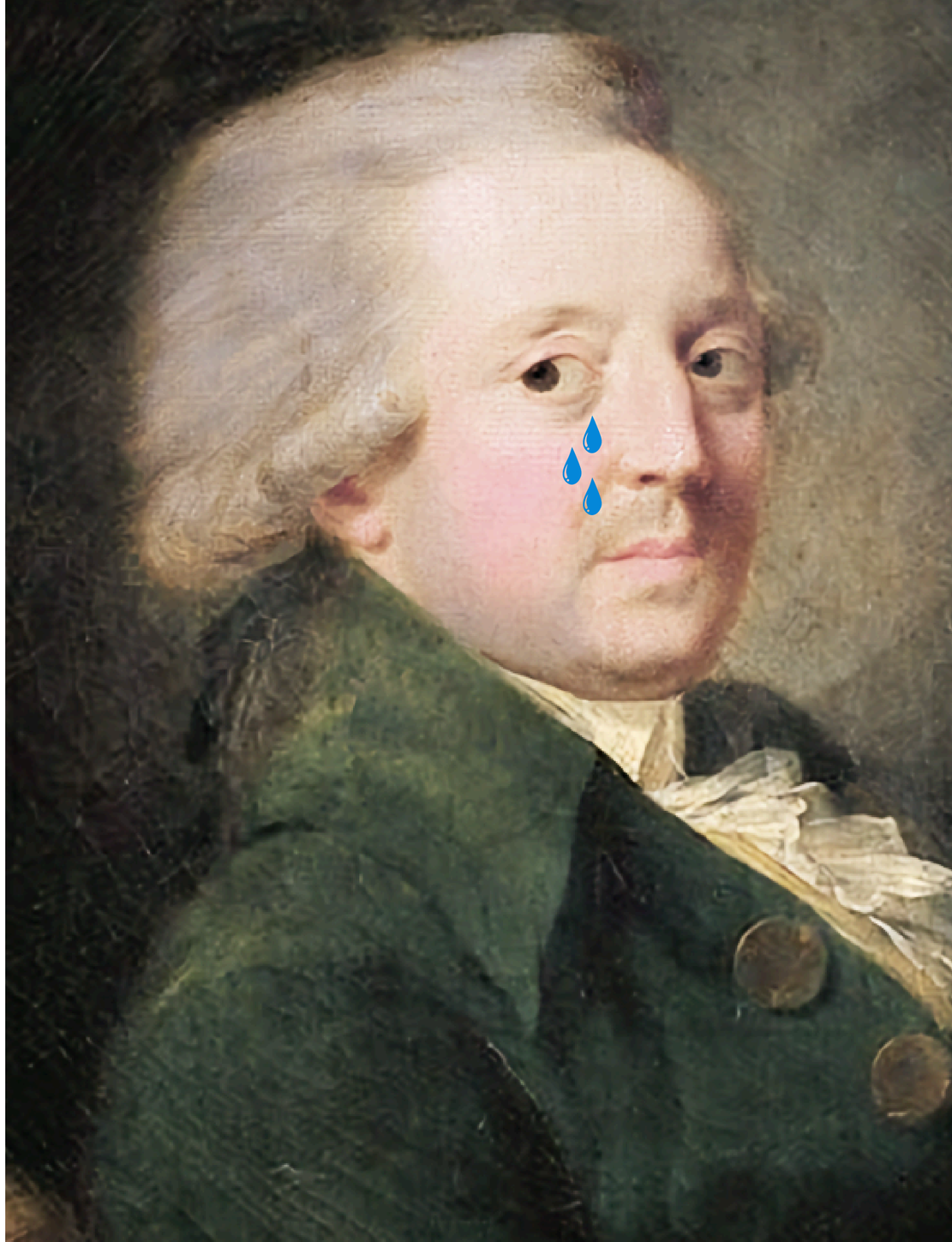
Advocate of universal human rights.

Fell afoul of the new revolutionary government and died in jail.

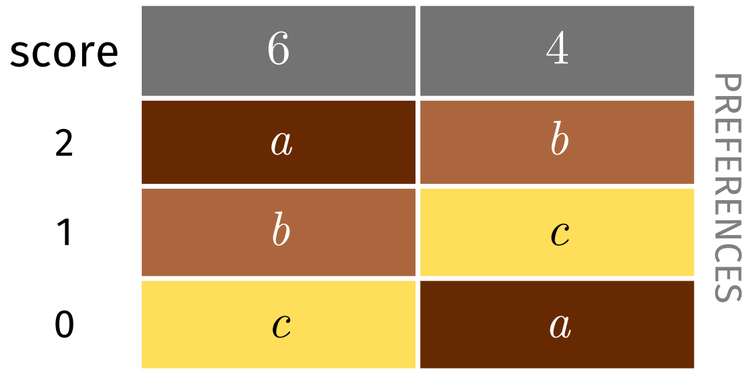


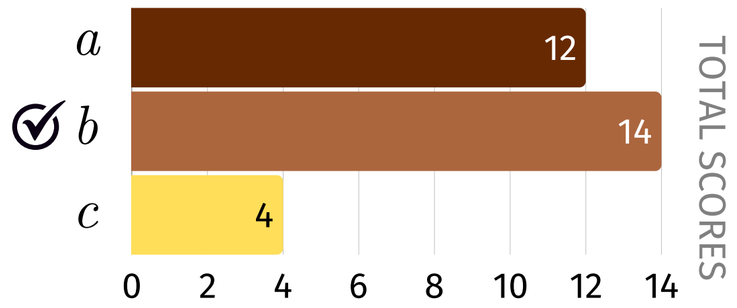
Recall that the Borda rule can choose a candidate whom a majority can deem unsuitable.

BORDA WINNER IS MAJORITY-DOMINATED BY ANOTHER CANDIDATE

Alternative *b* is the Borda winner.

But a majority prefers *a* to *b*.





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The winner of an election shouldn’t lose in a head-to-head contest with another alternative.

In fact, the winner should *win* all its head-to-head contests.

DEFINITION

A *Condorcet winner* is an alternative that wins in a head-to-head election with every other alternative.

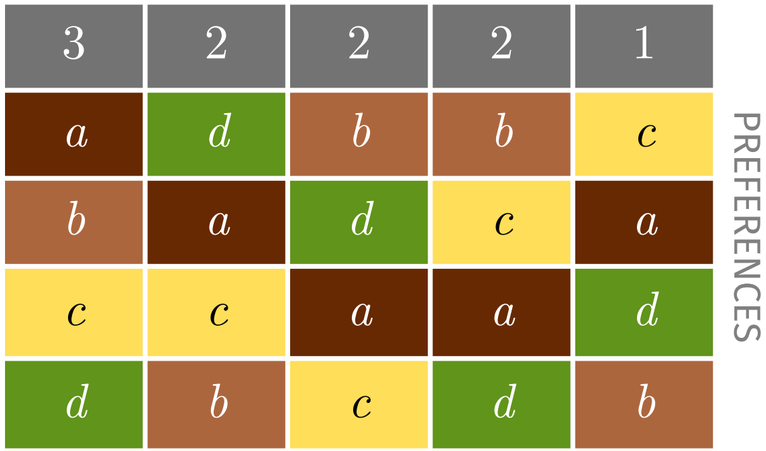
WHAT'S THE CONDORCET WINNER?

a wins over *b*: 6 votes to 4.

a wins over *c*: 7 votes to 3.

a wins over *d*: 6 votes to 4.

a is the Condorcet winner!



The Condorcet winner can be seen from the *majority graph*.

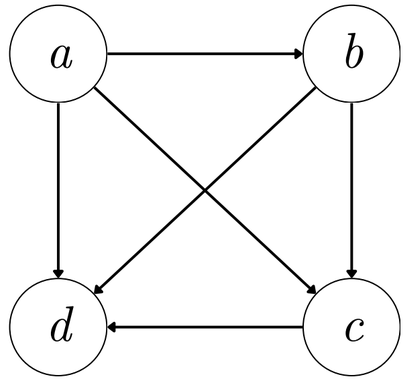
THE MAJORITY GRAPH

Draw an edge from *x* to *y* if a majority prefers *x* to *y*.

The Condorcet winner has only outgoing edges.

3	2	2	2	1
a	d	b	b	c
b	a	d	c	a
c	c	a	a	d
d	b	c	d	b

PREFERENCES



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Have people rank candidates and choose the Condorcet winner.

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BORDA

There's one small problem with that...

THE CONDORCET WINNER MAY NOT EXIST!

1	1	1
a	b	c
b	c	a
c	a	b

PREFERENCES

THE CONDORCET WINNER MAY NOT EXIST!

A majority prefers *a* to *b*.

A majority prefers *b* to *c*.

A majority prefers *c* to *a*.

What to do?

PREFERENCES

1	1	1
a	b	c
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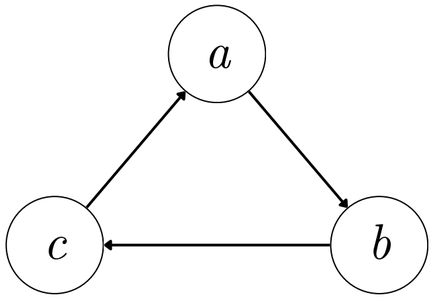
MAJORITY GRAPH



Perhaps we could declare all the candidates in a majority cycle as tied winners?

RESOLVING CYCLES TO TIES DOESN'T WORK

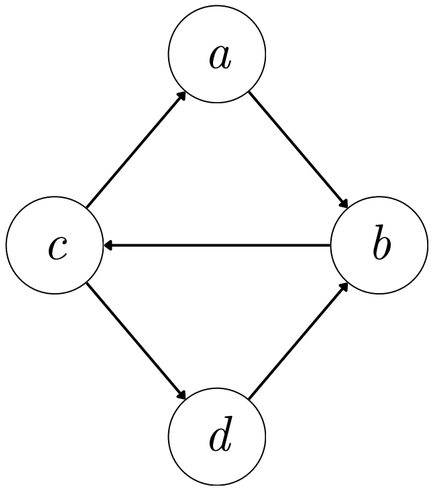
There are two cycles: $a \to b \to c \to a$, and $b \to c \to d \to b$.

Resolving the first cycle makes a, b and c tied.

Resolving the second cycle makes b, c and d tied.

By transitivity, a should be d tied.

But a majority can prefer a to d!



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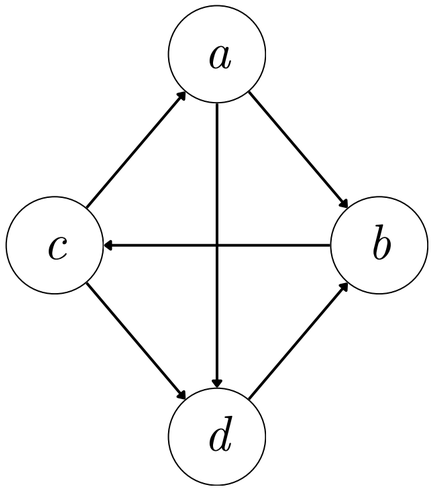


THE MARQUIS DE CONDORCET

Ok, but apart from the non-existence problem.

We should still select the Condorcet winner *when* it exists...

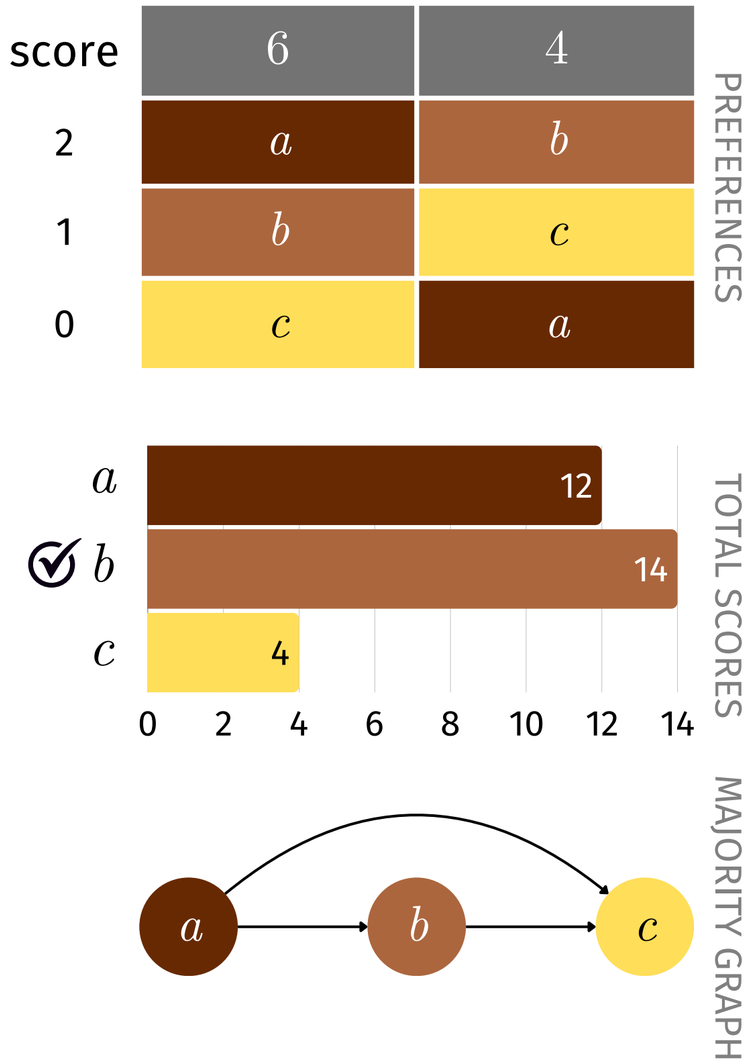
DEFINITION

A *Condorcet consistent* voting rule selects the Condorcet winner, if it exists.

THE BORDA RULE IS NOT CONDORCET CONSISTENT

Alternative *b* is the Borda winner.

But *a* is the Condorcet winner.



By the way, how badly can the Borda winner do with respect to the Condorcet rule?

As in, how many head-to-head contests can an alternative *lose* and still be the Borda winner?

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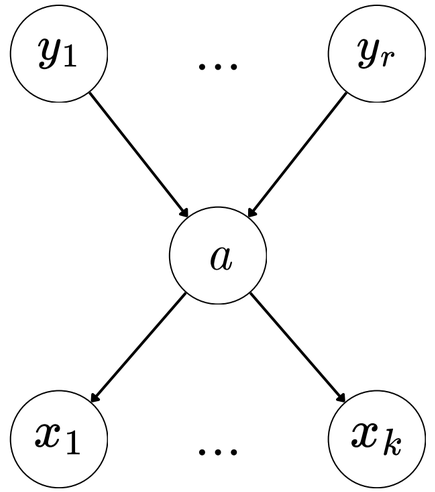
Interesting to think about for a project...

THE LLULL (COPELAND) RULE IS CONDORCET CONSISTENT (?)

Recall that Llull's (Copeland's) rule ranks alternatives according to the difference between the number of head-to-head wins and losses.

If *a* is the Condorcet winner, it wins all its head-to-head contests.



Llull (Copeland) score of *a*:

$$k - r$$

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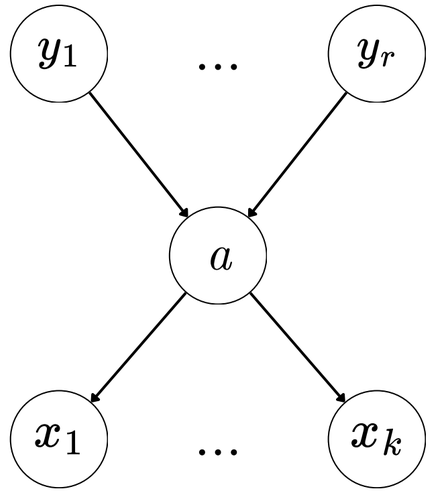
If *a* is the Condorcet winner, it wins all its head-to-head contests.

So it also has the maximal Llull (Copeland) score.

And hence is the winner according to this rule.



Llull (Copeland) score of *a*:

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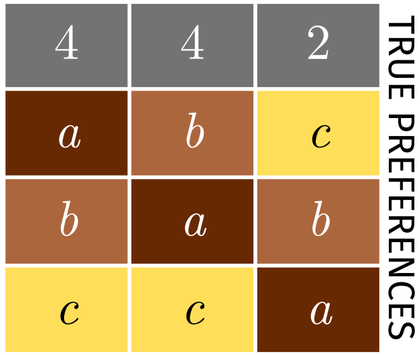
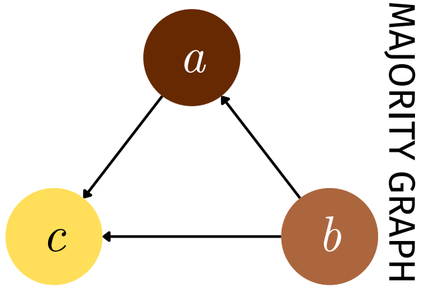
What about tactical voting?

MANIPULATING A CONDORCET CONSISTENT RULE

Let’s assume that majority cycles are broken lexicographically, i.e., *a* gets chosen before *b*, *b* before *c*.

b is the Condorcet winner.





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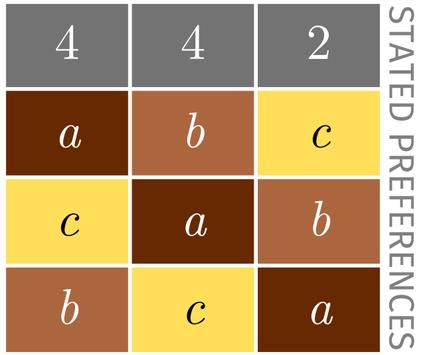
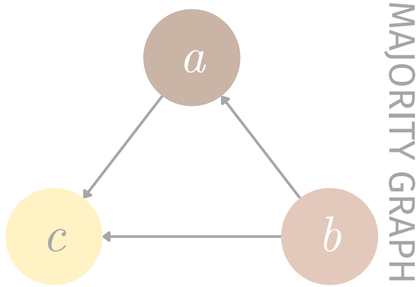
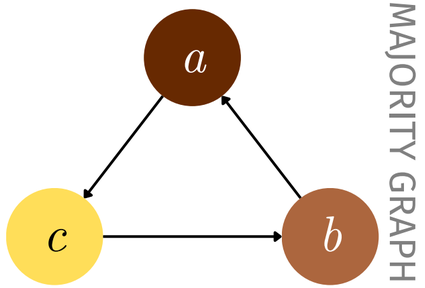
Now the first group of voters can manipulate by burying *c*.

In doing so a cycle is created, which gets resolved in favor of *a*.







There is, however, a way to circumvent this.

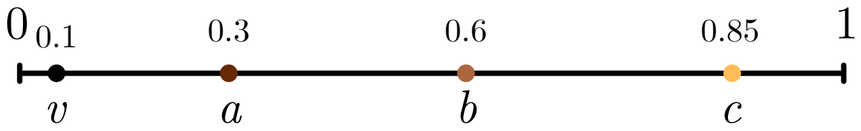
SPATIAL VOTING

Assume voters and alternatives are embedded in some type of space.

Think: ideological space.

Voters rank alternatives depending on how close they are.

Voter *v* thinks *a* is best, followed by *b*, followed by *c*.



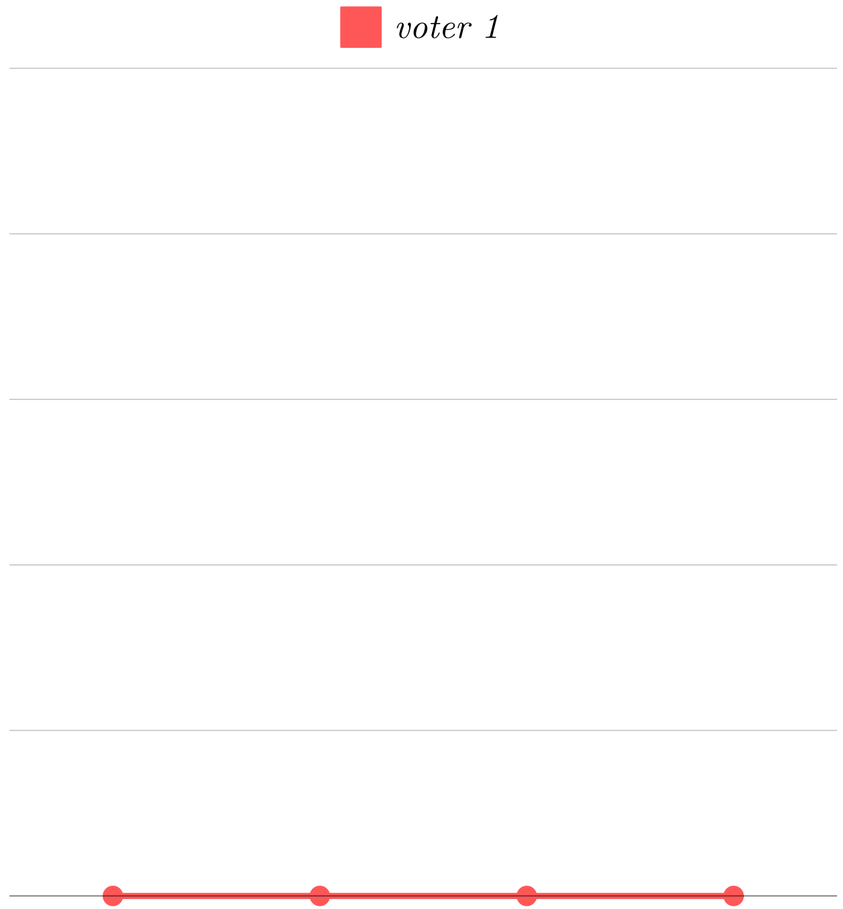
This can be thought of as preferences having a specific type of *structure*.

DEFINITION

A profile is *single-peaked* if:

1. alternatives can be ordered linearly, e.g., from left to right, and
2. every voter has a most preferred alternative, with other alternatives less preferred the further away they are to the ideal.

$\succ_1$	$\succ_2$	$\succ_3$
b	c	a
a	d	b
c	b	c
d	a	d



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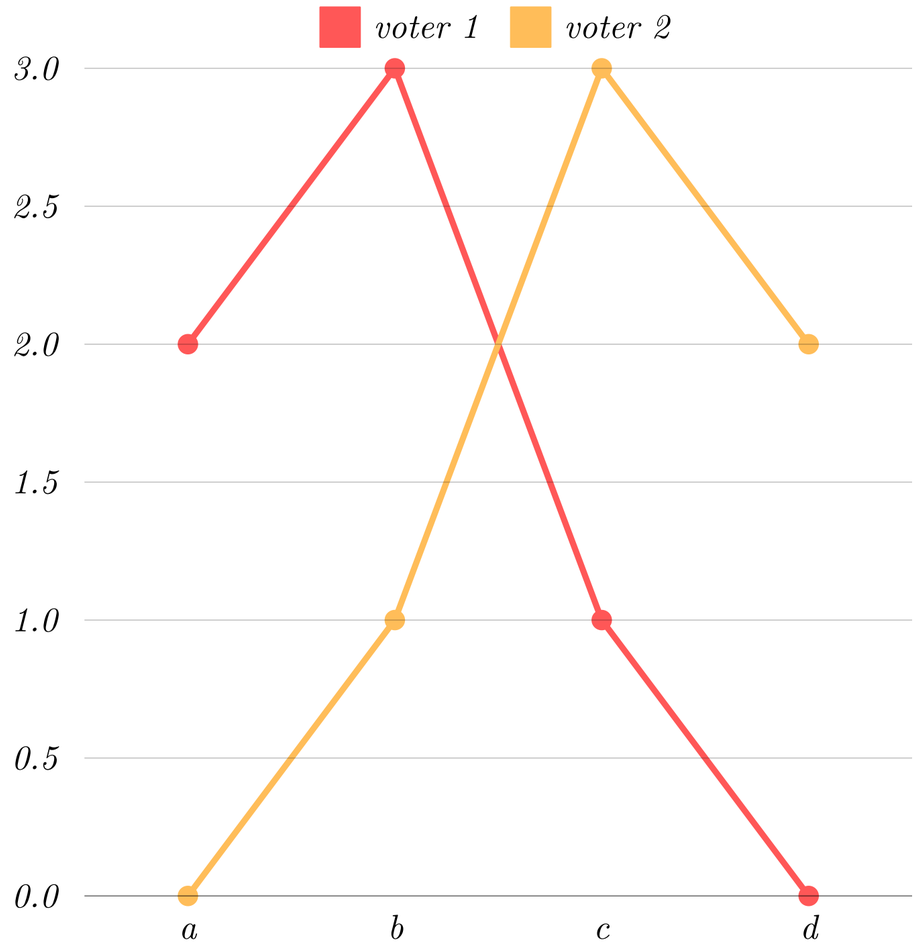

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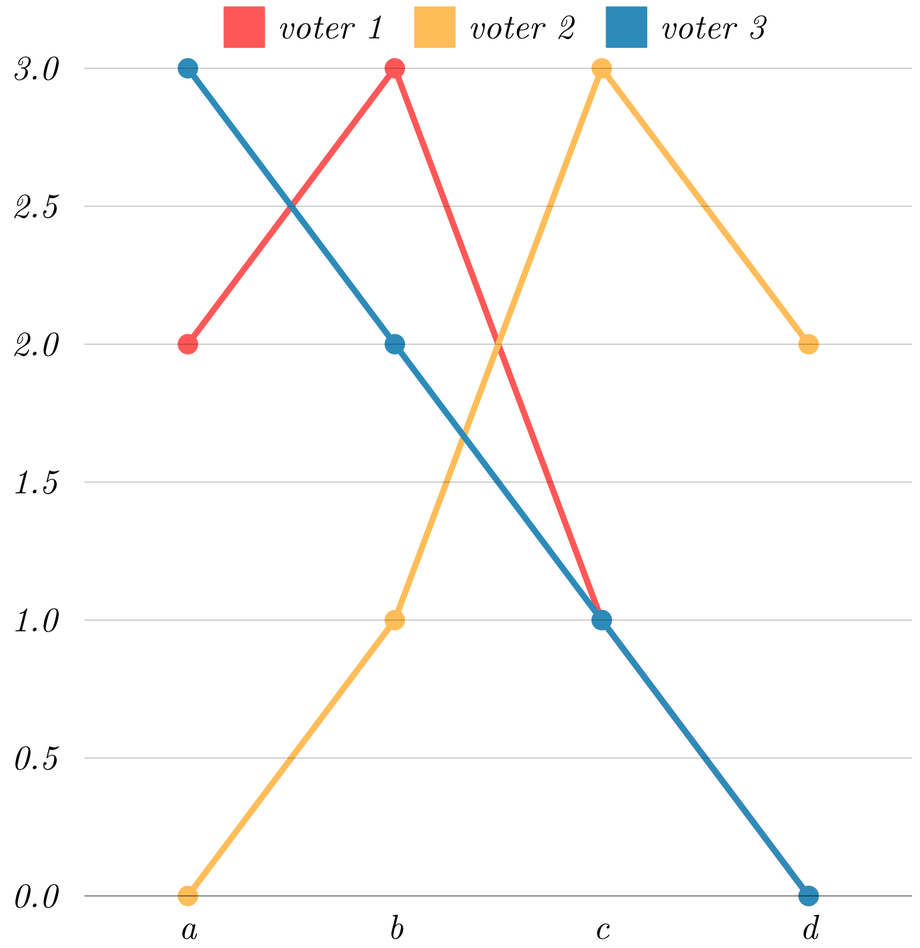
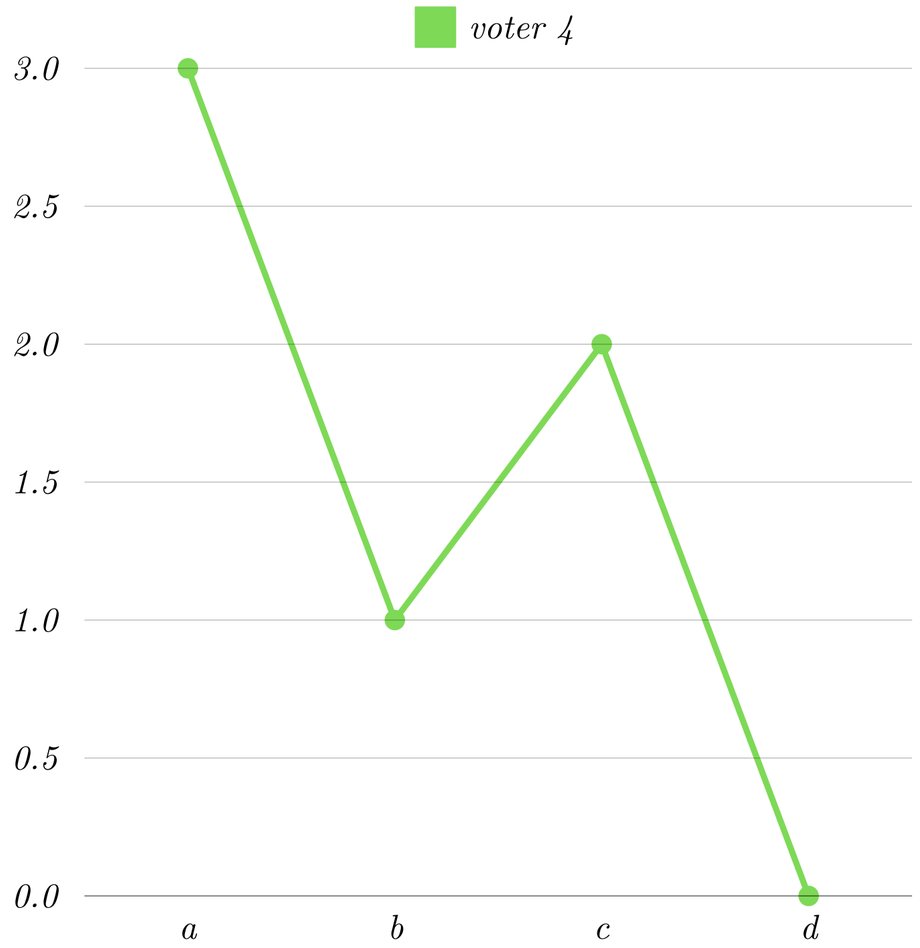


$\succ_4$
a
c
b
d



$\succ_4$
a
c
b
d

NOT SINGLE-PEAKED

MEDIAN VOTER THEOREM

THEOREM (BLACK, 1948)

For an odd number of voters, if the profile is single-peaked then the median peak is a Condorcet winner.

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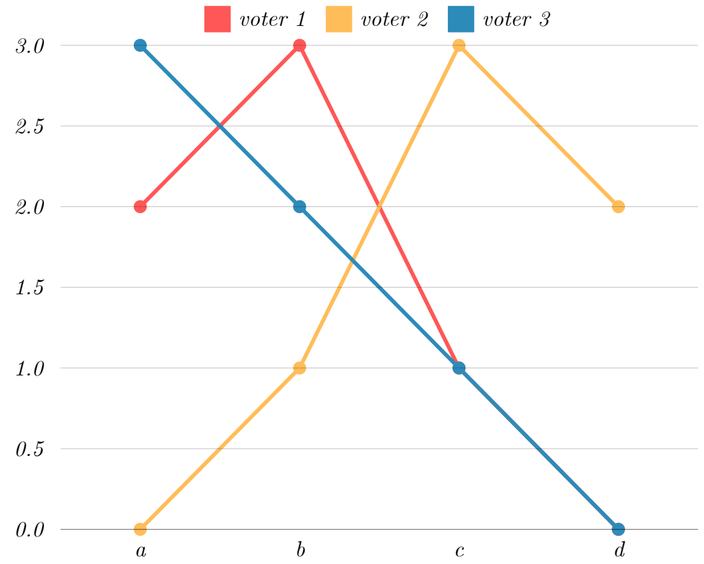

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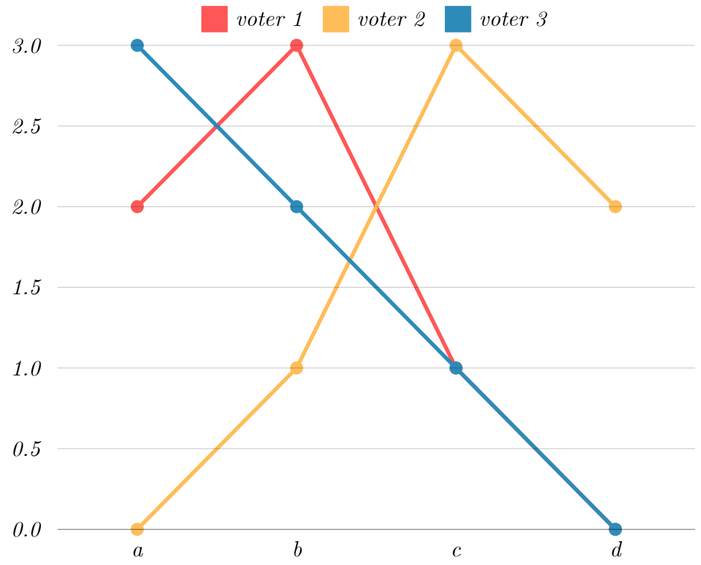
PROOF

If alternative *x* is the median peak, all voters whose peak is to the right of, and including, *x* rank *x* higher than alternatives to its left. And there is a strict majority of such voters.



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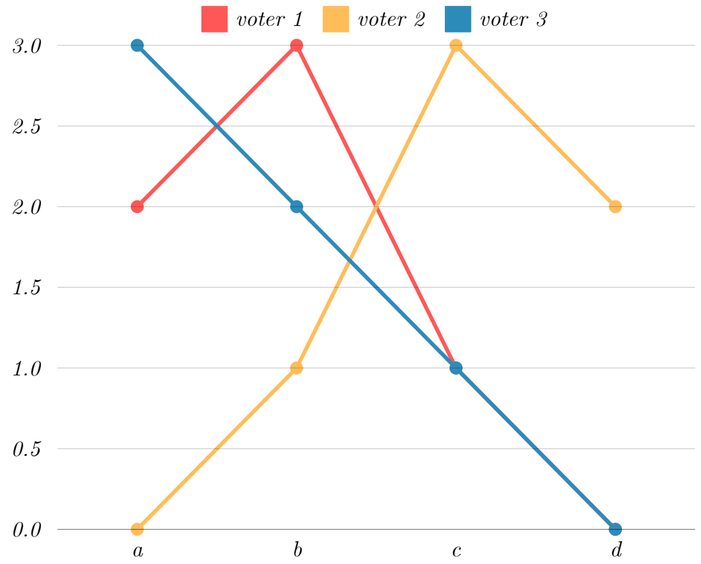

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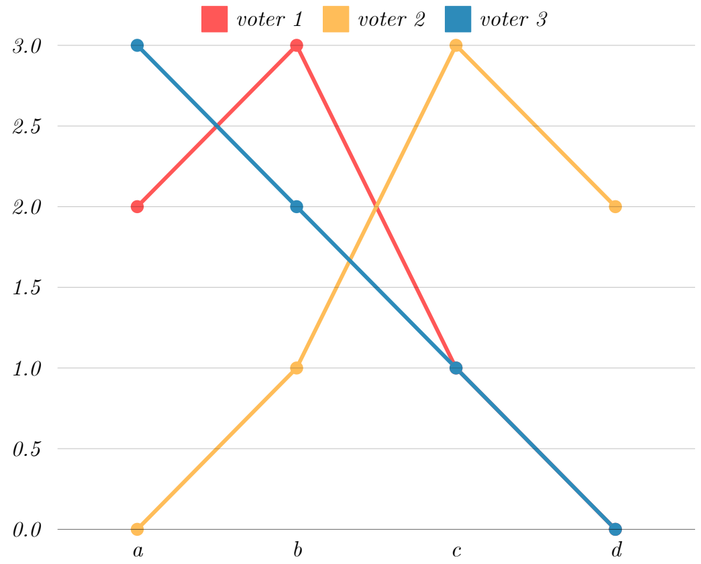
Similarly, voters whose peak is the left of, and including, x rank x higher than alternatives to its left.

Thus, x beats every other alternative in a head-to-head contest, i.e., is a Condorcet winner.



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MEDIAN VOTER THEOREM + EXTRA TOPPING

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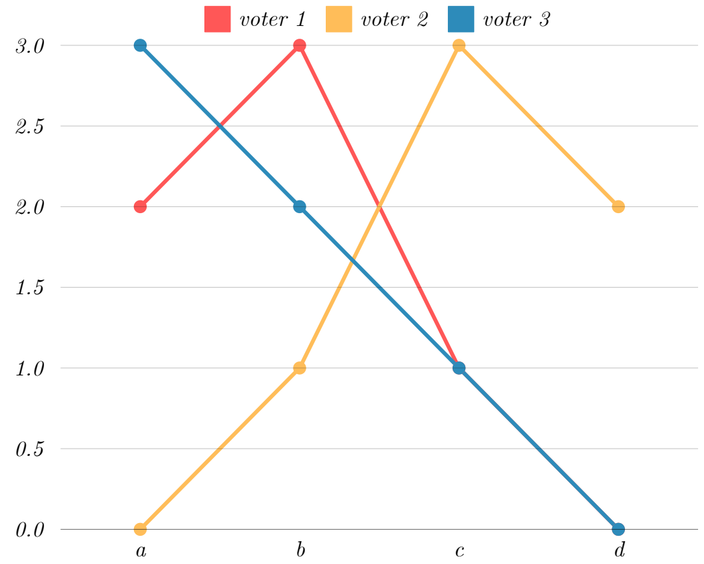

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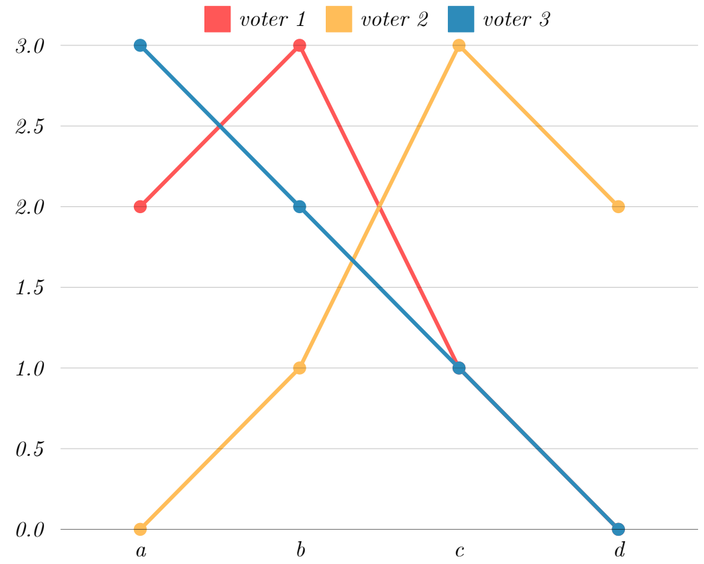

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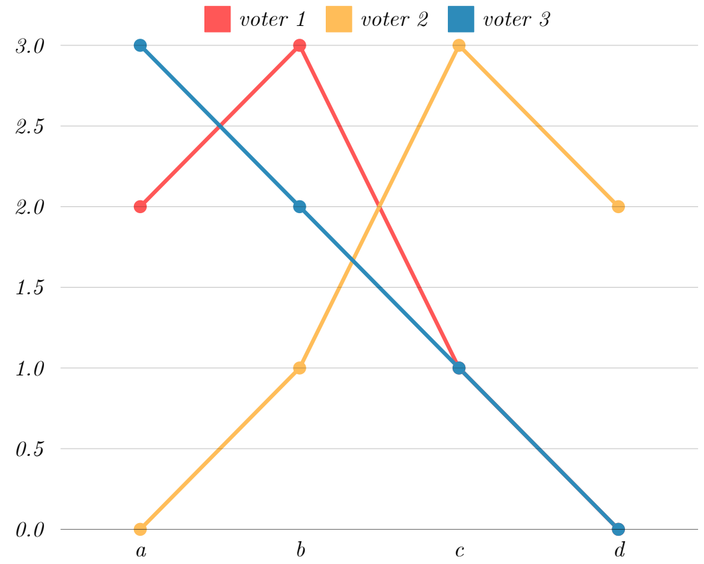
PROOF

Note that the median voter has no incentive to report a different ranking.

And if any other voter changes the median peak (by reporting a different ranking), this can only lead to a worse winner (for them)!



median peak: *b*

Supposing alternatives can also change their positions in order to atract more followers, where does this take us?...