Strategic Minds: The Game Theory of Cooperation, Coordination and Collaboration

Cooperate, or else

Repeated games, with and without discounting

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MERRIL FLOOD

Melvin and I came up with the idea behind the Prisoner's Dilemma in the 50's, while working for the RAND corporation.

MELVIN DRESHER

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For all the confusion, mutual cooperation occurred 60 out of the 100 trials.

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Game	AA	JW	AA's comments	JW's comments
1	D	C	JW will play [D]—sure win. Hence if I play [C]—I lose.	Hope he's bright.
2	D	C	What is he doing?!!	He isn't but maybe he'll wise up.
3	D	D	Trying mixed?	Okay, dope.
4	D	D	Has he settled on [D]?	Okay, dope.
5	C	D	Perverse!	It isn't the best of all possible worlds.
6	D	C	I'm sticking to [D] since he will mix for at least 4 more times.	Oh ho! Guess I'll have to give him another chance.
7	D	C		Cagey, ain't he? Well . . .
8	D	D		In time he could learn, but not in ten moves so:
9	D	D	If I mix occasionally, he will switch—but why will he ever switch from [D]?	
10	D	D	Prediction. He will stick with [D] until I change from [D]. I feel like DuPont.	I can guarantee myself a gain of 5, and guarantee that Player AA breaks

even (at best). On the other hand, with nominal assistance from AA, I can transfer the guarantee of 5 to Player AA and make 10 for myself, too. This means I have control of the game to a large extent, so Player AA had better appreciate this and get on the bandwagon.

With small amounts of money at stake, I would (as above) try (by using [C]) to coax AA into mutually profitable actions. With large amounts at stake I would play [D] until AA displayed some initiative and a willingness to invest in his own future. One play of [C] by AA would change me from [D] to [C], where I would remain until bitten.

On the last play, it would be conservative for me to switch to [D], but I wouldn't do so if the evidence suggested that AA was a nice stable personality

Game	AA	JW	AA's comments	JW's comments
				and not in critical need of just a little extra cash.
11	D	C		Probably learned by now.
12	C	C		I'll be damned! But I'll try again.
13	C	C		That's better.
14	C	C		Ha!
15	C	C		(bliss)
16	D	C		
17	C	D		The stinker.
18	C	D		He's crazy. I'll teach him the hard way.
19	D	D	I'm completely confused. Is he trying to convey information to me?	Let him suffer.
20	D	D		
21	D	C		Maybe he'll be a good boy now.
22	C	C		Always takes time to learn.

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Are AA and JW irrational?

MERRIL FLOOD

What do you say to that, John?!

JOHN NASH

•••

MERRIL FLOOD
What do you say to that, John?!

JOHN NASH
You know, playing the Prisoner’s Dilemma one time is not the same as playing it 100 times.

Playing it over and over again is like playing a different, multi-round game.

In the one-shot game there’s no room for things like loyalty, trust, threats, or revenge.

But in the iterated version, these things can be relevant!

This might give us a way out of the pessimistic outlook of the Prisoner’s Dilemma.

Does the equilibrium change if the game is played repeatedly?

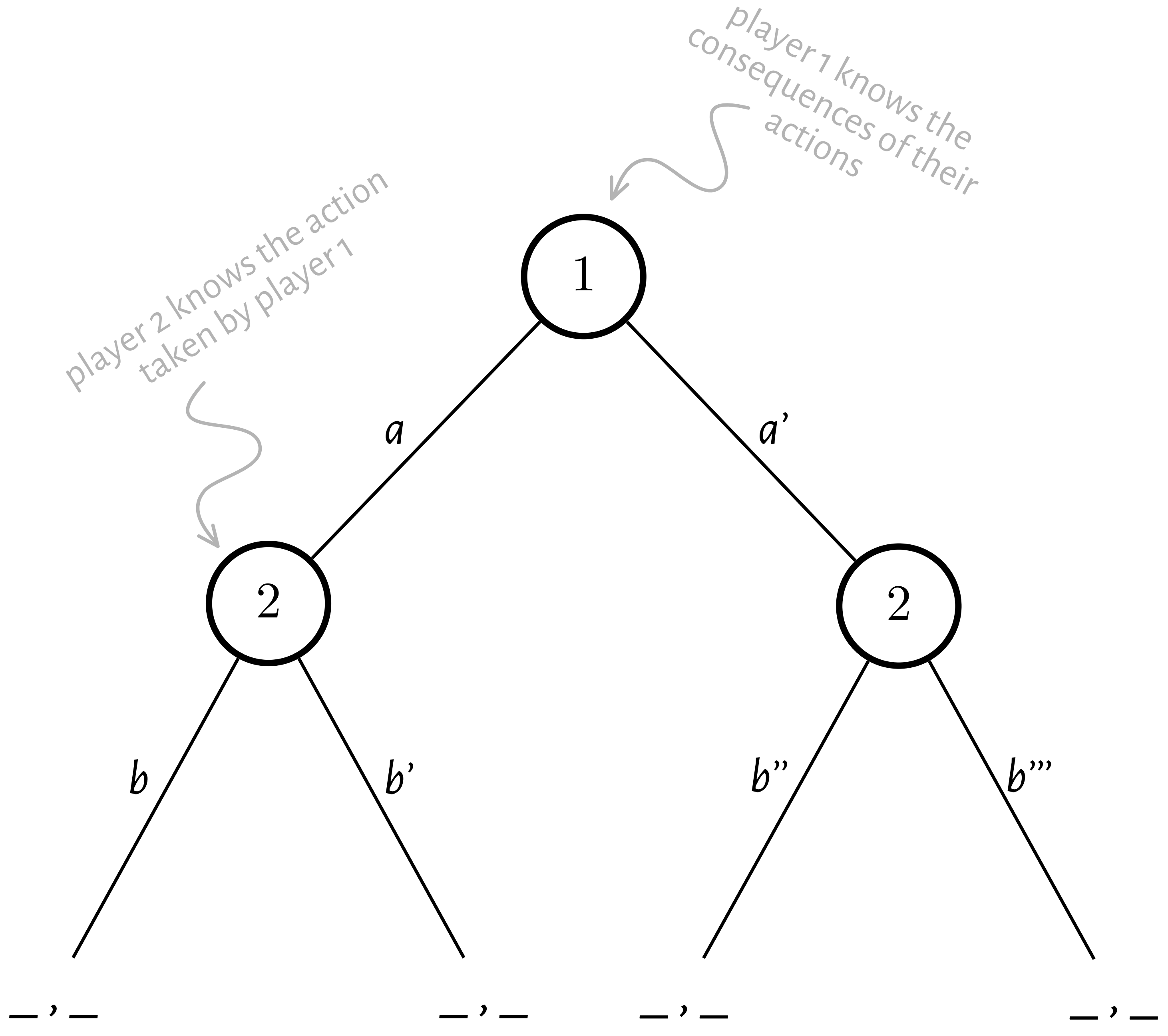
PERFECT-INFORMATION EXTENSIVE GAMES

So far we've been assuming that players make moves simultaneously, in ignorance of the other players' actions.

But, of course, some games are played over rounds.

In *perfect-information extensive-form games*, players take turns deploying their actions.

And are aware of actions taken at previous rounds: perfect memory!



The Ultimatum Game

Player 1 has two euros, which it has to divide between themselves and player 2.

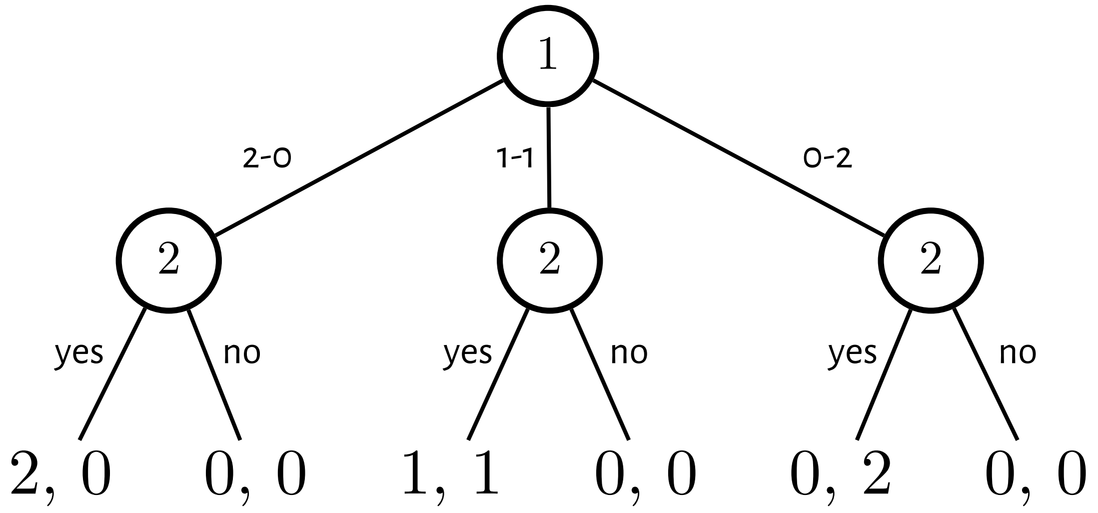
Player 1 makes an offer, which player 2 can *accept* or *reject*.

If player 2 accepts, money is divided according to player 1's offer.

If player 2 rejects, no one gets anything.

1/2

payoffs

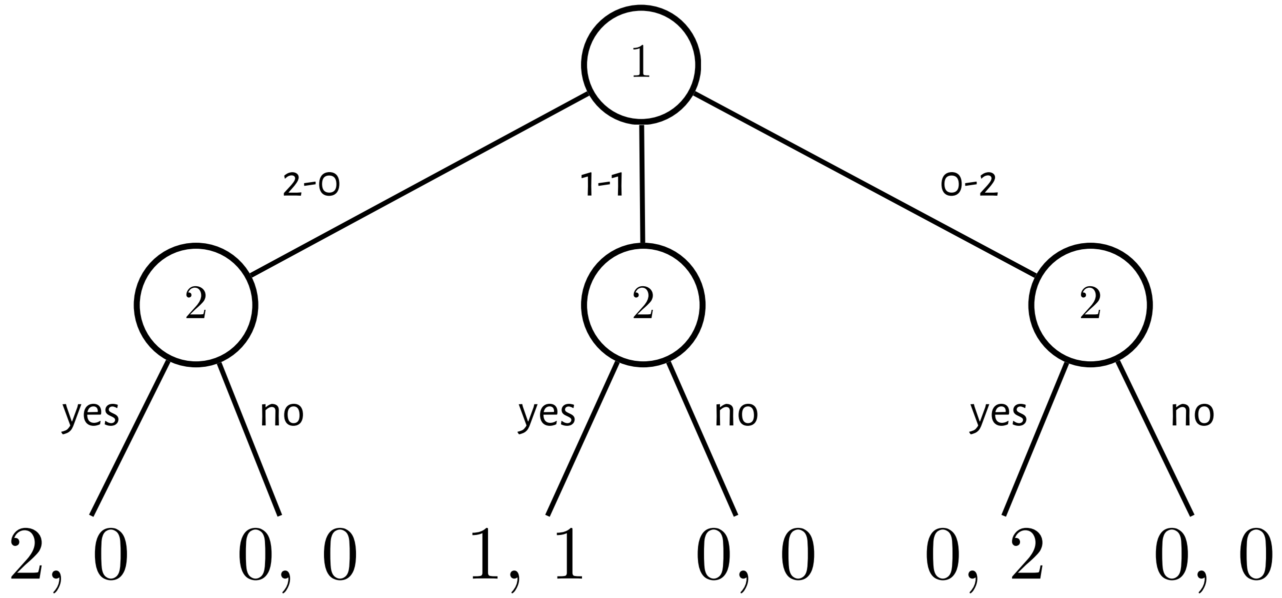


Pareto optimal strategy profiles

pure Nash equilibria

mixed Nash equilibria

2/2



	yyy	yyn	yny	ynn	nyy	nyn	nny	nnn
(2-0)	2, 0	2, 0	2, 0	2, 0	0, 0	0, 0	0, 0	0, 0
(1-1)	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0
(0-2)	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0

Nash equilibria and everything else is computed with respect to the induced normal-form game.

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payoffs

	yyy	yyn	yny	ynn	nyy	nyn	nny	nnn
(2-0)	2, 0	2, 0	2, 0	2, 0	0, 0	0, 0	0, 0	0, 0
(1-1)	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0
(0-2)	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0

Pareto optimal strategy profiles

?

pure Nash equilibria

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mixed Nash equilibria

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	yyy	yyn	yny	ynn	nyy	nyn	nny	nnn
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(1-1)	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0
(0-2)	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0

Pareto optimal strategy profiles

everything except (0, 0)

pure Nash equilibria

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mixed Nash equilibria

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	yyy	yyn	yny	ynn	nyy	nyn	nny	nnn
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(1-1)	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0
(0-2)	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0

Pareto optimal strategy profiles

everything except (0, 0)

pure Nash equilibria

see above

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payoffs

	yyy	yyn	yny	ynn	nyy	nyn	nny	nnn
(2-0)	2, 0	2, 0	2, 0	2, 0	0, 0	0, 0	0, 0	0, 0
(1-1)	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0
(0-2)	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0

Pareto optimal strategy profiles

everything except (0, 0)

pure Nash equilibria

see above

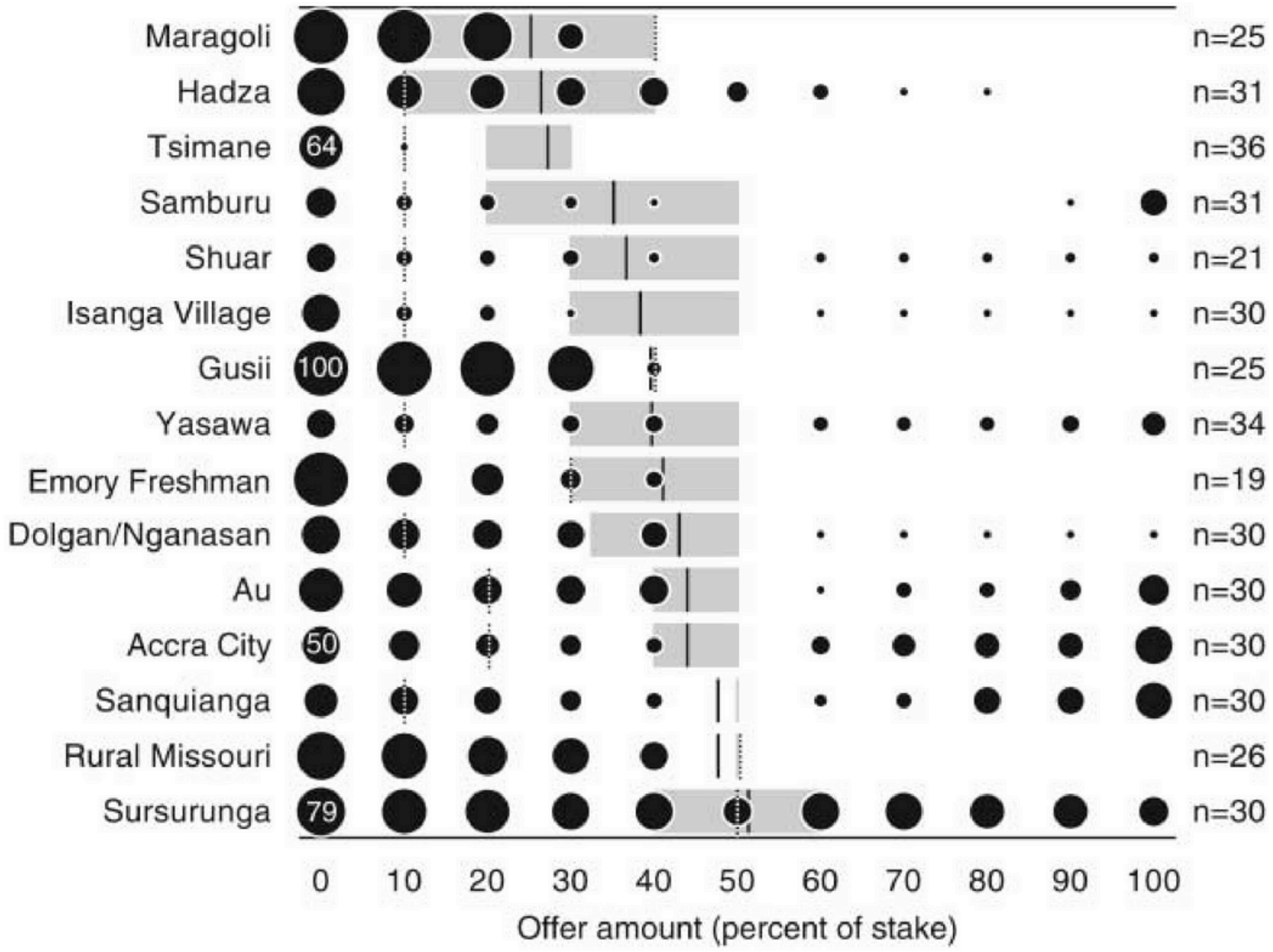
mixed Nash equilibria

too lazy to figure out

JOE HENRICH

There are interesting cultural differences in the offers people from different cultures accept and reject when playing The Ultimatum Game.

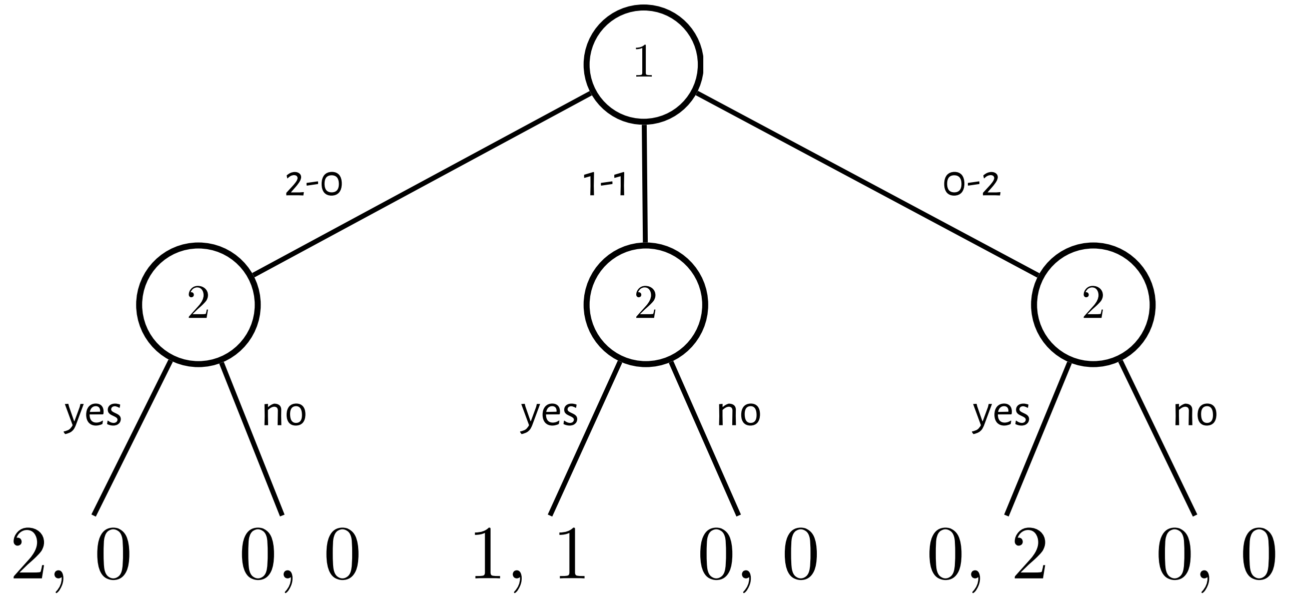
Henrich, J., McElreath, R., Barr, A., Ensminger, J., Barrett, C., Bolyanatz, A., Cardenas, J. C., Gurven, M., Gwako, E., Henrich, N., Lesorogol, C., Marlowe, F., Tracer, D., & Ziker, J. (2006). Costly punishment across human societies. Science, 312(5781), 1767–1770.



What makes (2-0, nnn) a Nash equilibrium depends crucially on what Player 2 does at *all* nodes: including 'irrelevant' ones.

Think: why does Player 1 not want to deviate?

Because Player 2 always says *no*, so there's no point!



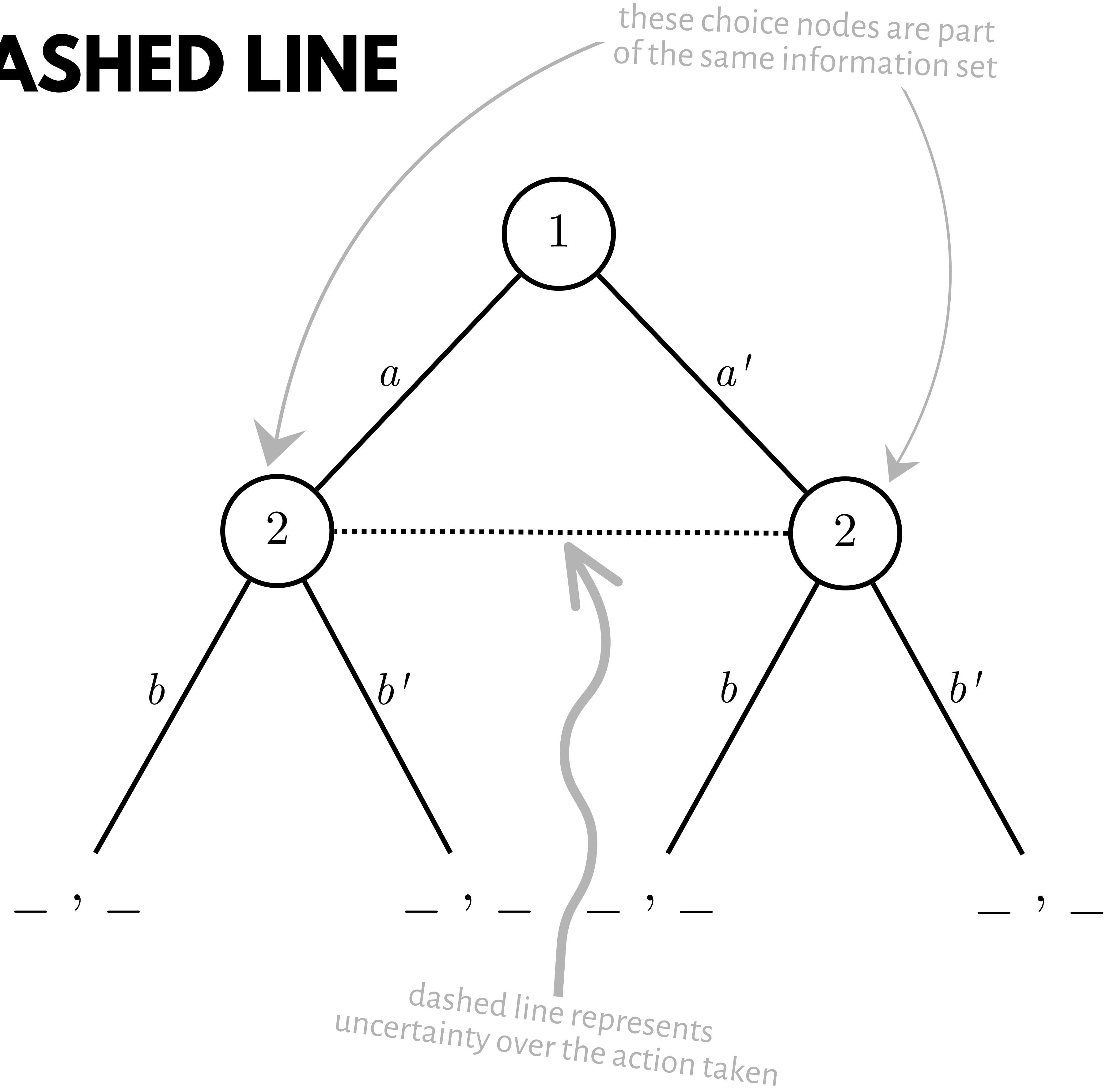
	yyy	yyn	yny	ynn	nyy	nyn	nny	nnn
(2-0)	2, 0	2, 0	2, 0	2, 0	0, 0	0, 0	0, 0	0, 0
(1-1)	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0
(0-2)	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0

In general, we can always transform an extensive-information game with perfect information into a game in normal form.

Enter extensive-form games with *imperfect* information.

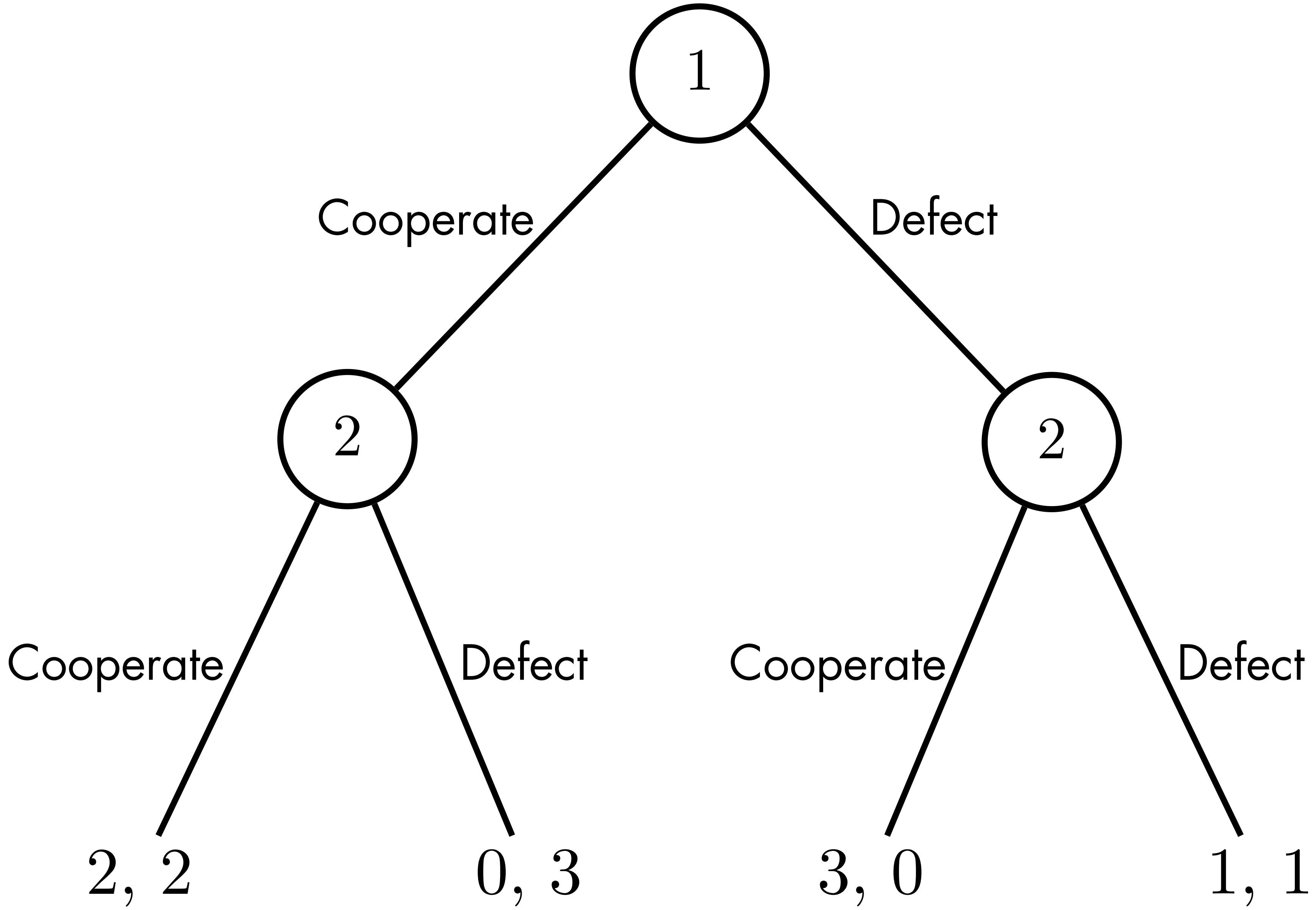
ADDING UNCERTAINTY: A DASHED LINE

Player 2 does *not* know what action Player 1 has actually taken.



Now we can finally get back to the Prisoner's Dilemma!

	Cooperate	Defect
Cooperate	2, 2	0, 3
Defect	3, 0	1, 1



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Cooperate	2, 2	0, 3
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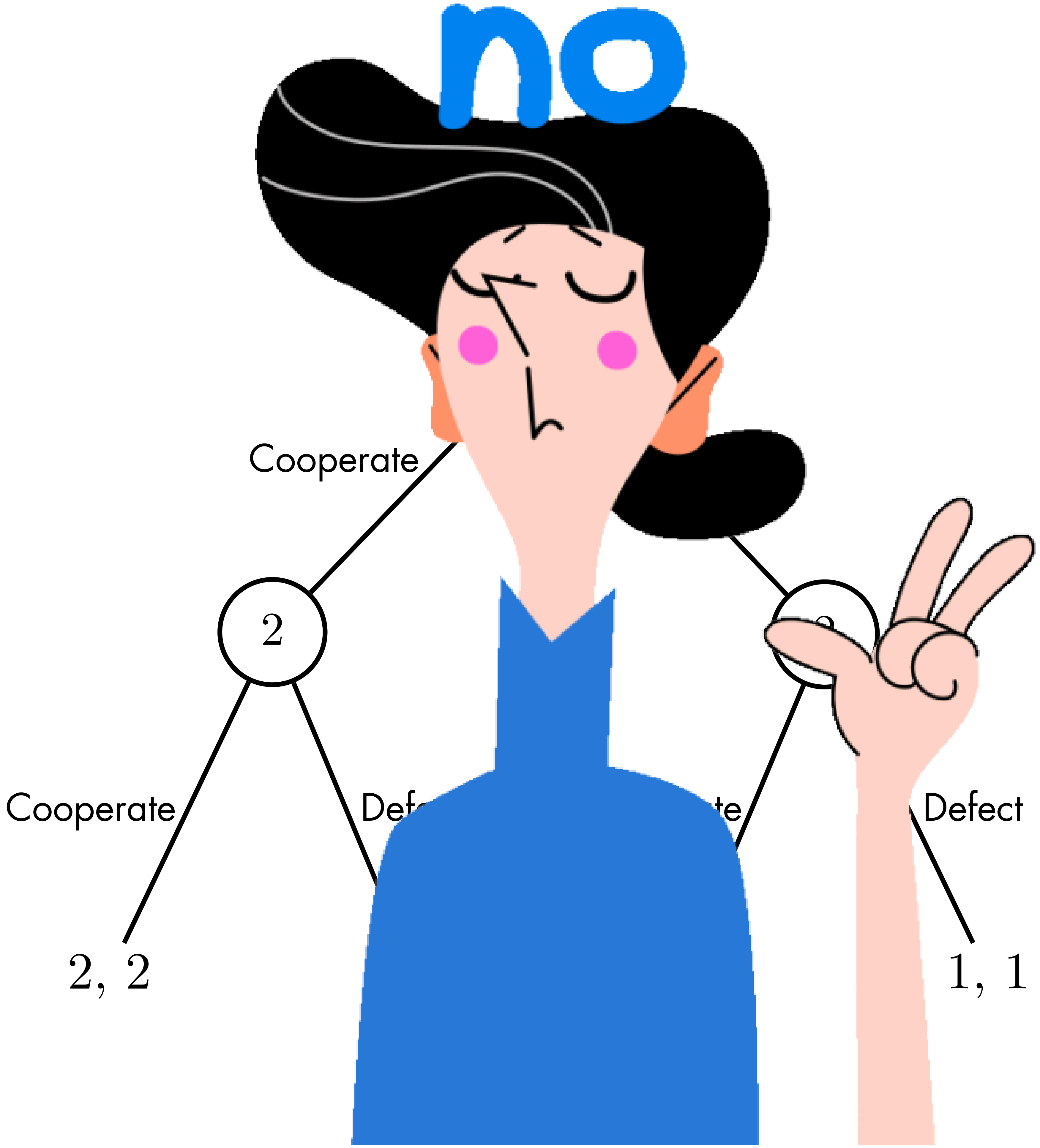

	Cooperate	Defect
Cooperate	2, 2	0, 3
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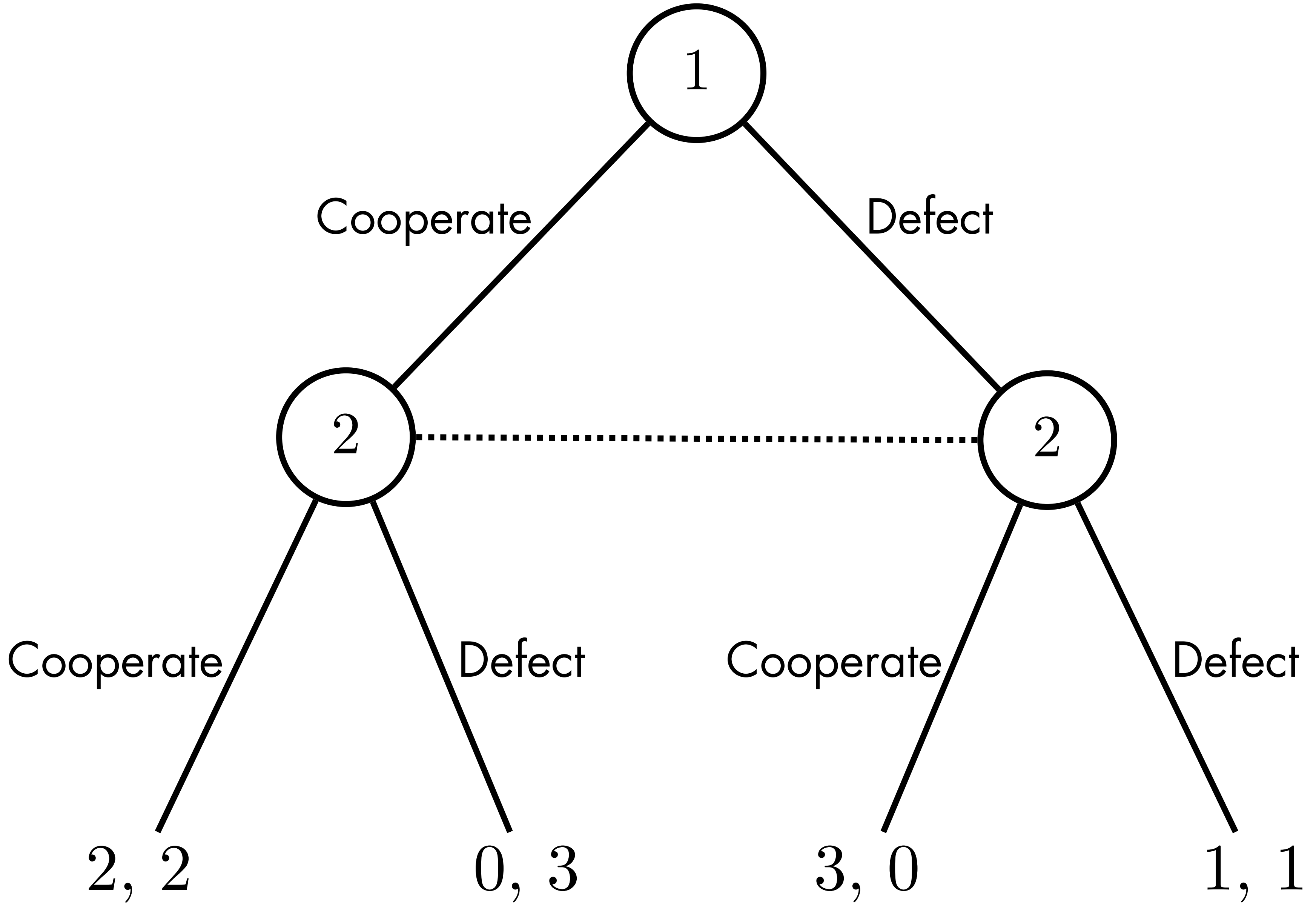
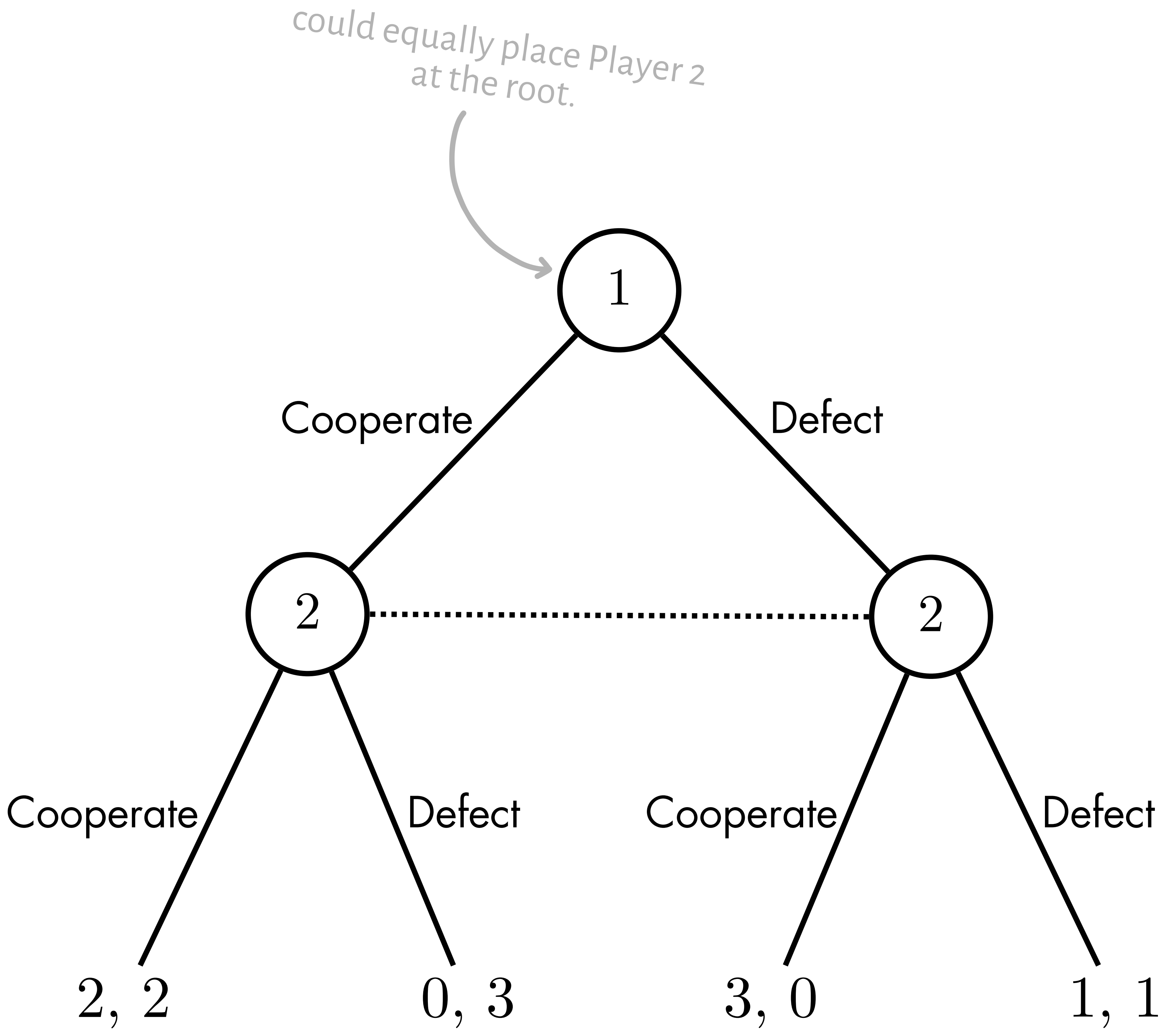


	Cooperate	Defect
Cooperate	2, 2	0, 3
Defect	3, 0	1, 1



Now we can even model the iterated Prisoner’s Dilemma!

A finite number of rounds.

Like, say, two.

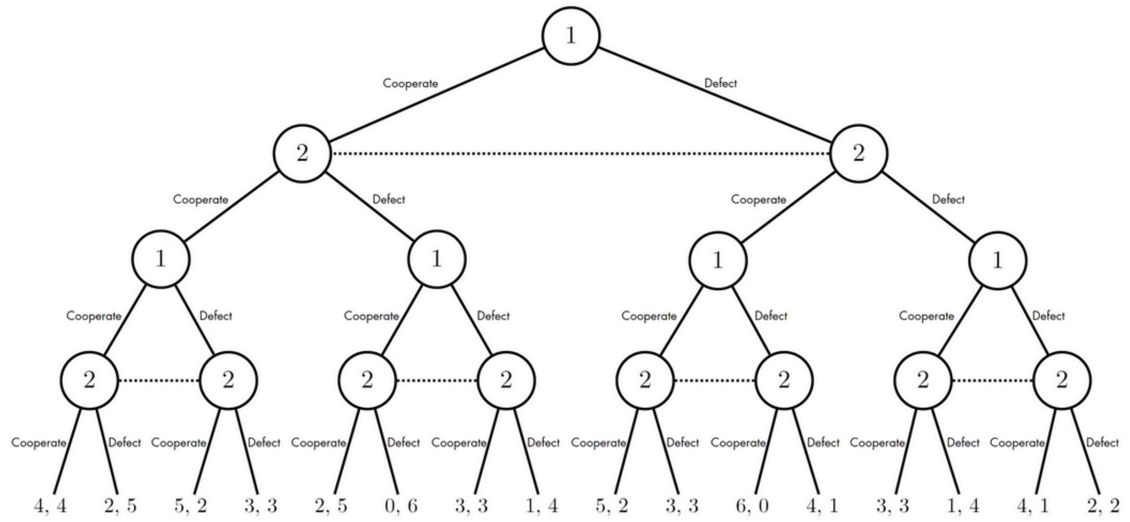
Iterated Prisoner's Dilemma 

2 iterations

Two players play the Prisoner's Dilemma over k = 2 rounds.

The final payoffs are the sum of the payoffs from each round.

payoffs



strictly dominant strategies

?

Pareto optimal strategy profiles

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pure Nash equilibria

?

mixed Nash equilibria

?

Iterated Prisoner’s Dilemma 

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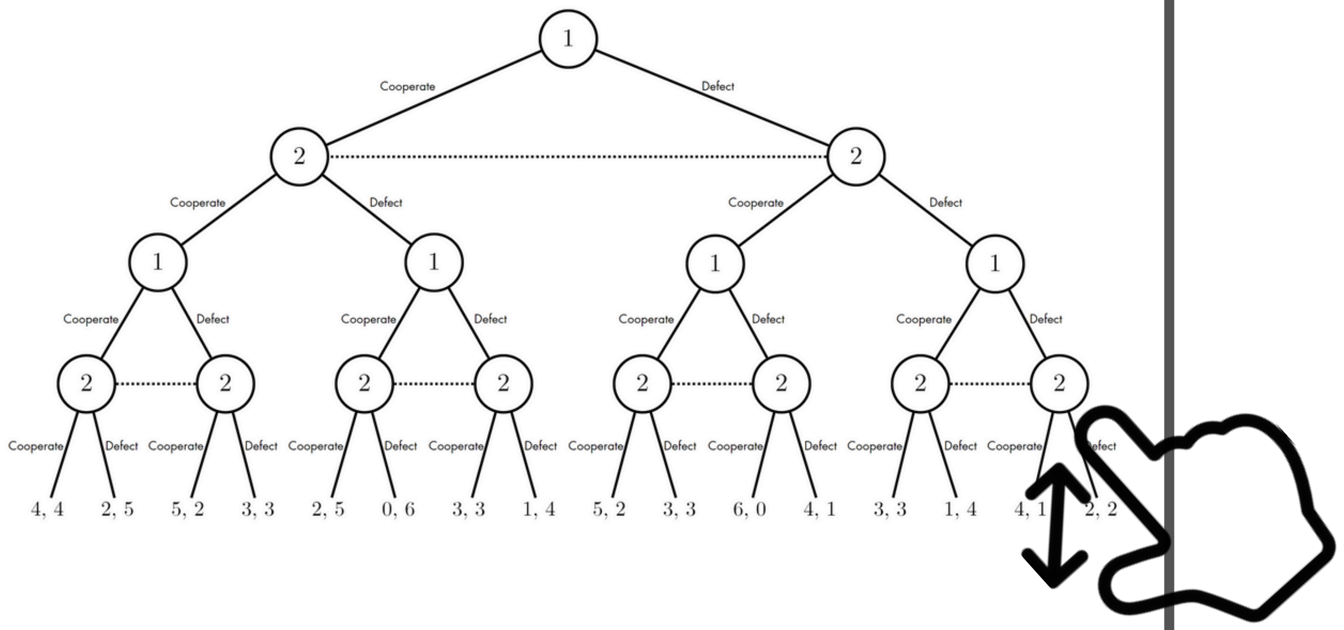

strictly dominant strategies

?

Pareto optimal strategy profiles

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pure Nash equilibria

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mixed Nash equilibria

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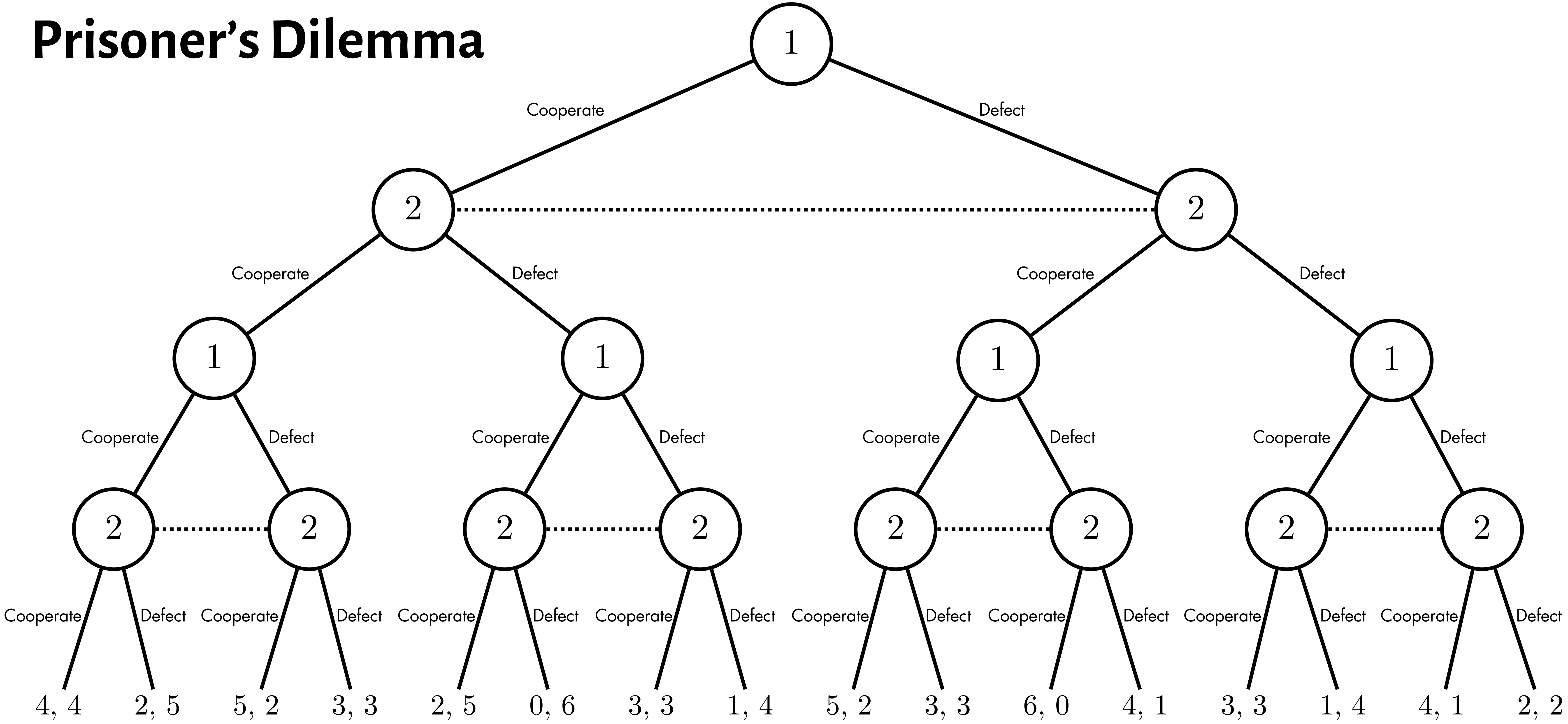
Two Rounds of the Prisoner's Dilemma



Iterated Prisoner's Dilemma 

2 iterations

Two players play the Prisoner's Dilemma over k = 2 rounds.

The final payoffs are the sum of the payoffs from each round.

payoffs

	C, C	C, D	D, C	D, D
C, C	$2+2, 2+2$	$2+0, 2+3$	$0+2, 3+2$	$0+0, 3+3$
C, D	$2+3, 2+0$	$2+1, 2+1$	$0+3, 3+0$	$0+1, 3+1$
D, C	$3+2, 0+2$	$3+0, 0+3$	$1+2, 1+2$	$1+0, 1+3$
D, D	$3+3, 0+0$	$3+1, 0+1$	$1+3, 1+0$	$1+1, 1+1$

strictly dominant strategies

?

Pareto optimal strategy profiles

?

pure Nash equilibria

?

mixed Nash equilibria

?

Iterated Prisoner's Dilemma

2 iterations

Two players play the Prisoner's Dilemma over $k = 2$ rounds.

The final payoffs are the sum of the payoffs from each round.

payoffs

	C, C	C, D	D, C	D, D
C, C	4, 4	2, 5	2, 5	0, 6
C, D	5, 2	3, 3	3, 3	1, 4
D, C	5, 2	3, 3	3, 3	1, 4
D, D	6, 0	4, 1	4, 1	2, 2

strictly dominant strategies

?

Pareto optimal strategy profiles

?

pure Nash equilibria

?

mixed Nash equilibria

?

So how do we analyze the 2-round Prisoner's Dilemma?

Iterated Prisoner's Dilemma 

2 iterations

Two players play the Prisoner's Dilemma over k = 2 rounds.

The final payoffs are the sum of the payoffs from each round.

payoffs

	C, C	C, D	D, C	D, D
C, C	4, 4	2, 5	2, 5	0, 6
C, D	5, 2	3, 3	3, 3	1, 4
D, C	5, 2	3, 3	3, 3	1, 4
D, D	6, 0	4, 1	4, 1	2, 2

Pareto optimal strategy profiles

see above

pure Nash equilibria

?

mixed Nash equilibria

Iterated Prisoner's Dilemma

2 iterations

Two players play the Prisoner's Dilemma over k = 2 rounds.

The final payoffs are the sum of the payoffs from each round.

payoffs

	C, C	C, D	D, C	D, D
C, C	4, 4	2, 5	2, 5	0, 6
C, D	5, 2	3, 3	3, 3	1, 4
D, C	5, 2	3, 3	3, 3	1, 4
D, D	6, 0	4, 1	4, 1	2, 2

Pareto optimal strategy profiles

see previous

pure Nash equilibria

((D, D), (D, D))

mixed Nash equilibria

?

Iterated Prisoner's Dilemma 

2 iterations

Two players play the Prisoner's Dilemma over k = 2 rounds.

The final payoffs are the sum of the payoffs from each round.

payoffs

	C, C	C, D	D, C	D, D
C, C	4, 4	2, 5	2, 5	0, 6
C, D	5, 2	3, 3	3, 3	1, 4
D, C	5, 2	3, 3	3, 3	1, 4
D, D	6, 0	4, 1	4, 1	2, 2

Pareto optimal strategy profiles

see previous

pure Nash equilibria

((D, D), (D, D))

mixed Nash equilibria

none

Again, the only Nash equilibrium is to always defect, for both players.

Note that we'd get the same conclusion for $k > 2$ rounds.

Well that was pointless.

Quick recap.

In the Prisoner’s Dilemma, the unique Nash equilibrium requires both players to defect.

We often observe cooperation in the real world.

What should we add to our model to make cooperation rational?

Maybe if players acknowledge they are in a repeated relationship.

Unfortunately, if the Prisoner’s Dilemma is repeated a commonly known finite number of times, the Nash equilibrium is still defect at every round.

ROBERT AUMANN

What if the game is played for an infinite number of times?

As in, we don't have a fixed number *k* of rounds at which the game ends.

Iterated Prisoner's Dilemma

infinitely iterated

Two players play the regular Prisoner's Dilemma:

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

but an infinite number of times.

The final payoffs are the sum of the payoffs from each round.

Players

$N = \{1, 2\}$

Strategies of Player 1

(C, C, ...), (C, D, ...), ...

Strategies of Player 2

(C, C, ...), (C, D, ...), ...

Payoffs (aka utilities)

In general, infinite sums.

For instance, if both players always cooperate, payoffs are infinite series: (2, 2, ...), and the final payoff is:

$$2 + 2 + \cdots = \infty$$

ROBERT AUMANN

Let's also add a *discount factor* δ, with 0 < δ < 1, which works as follows.

At every new round, the payoffs are multiplied by δ.

ROBERT AUMANN

Let's also add a *discount factor* δ, with $0 < \delta < 1$, which works as follows.

At every new round, the payoffs are multiplied by δ.

So for $\delta = 0.8$, \$100 today is worth $0.8 \cdot \$100 = \$80$ tomorrow, and $0.8 \cdot \$80 = \$64$ in two days.

Iterated Prisoner's Dilemma

infinitely iterated, with discount factor, $0 < \delta < 1$

Two players play the regular Prisoner's Dilemma:

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

but an infinite number of times.

The final payoffs are the sum of the payoffs from each round, taking into account the discount factor δ.

1/2

Players

$N = \{1, 2\}$

Strategies of Player 1

(C, C, ...), (C, D, ...), ...

Strategies of Player 2

(C, C, ...), (C, D, ...), ...

Payoffs (aka utilities)

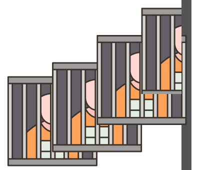
In general, infinite sums.

For instance, if both players always cooperate, payoffs are infinite series: $(2, 2\delta, 2\delta^2, \ldots)$, and the final payoff is:

$$2 + 2\delta + 2\delta^2 + \ldots$$

In general, for infinite sums we can use the following identity, for $0 < x < 1$:

$$1 + x + x^2 + \cdots = \frac{1}{1-x}$$

Iterated Prisoner's Dilemma 

infinitely iterated, with discount factor, $0 < \delta < 1$

Two players play the regular Prisoner's Dilemma:

	C	D
C	2, 2	0, 3
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but an infinite number of times.

The final payoffs are the sum of the payoffs from each round, taking into account the discount factor δ.

1/2

Players

$N = \{1, 2\}$

Strategies of Player 1

(C, C, ...), (C, D, ...), ...

Strategies of Player 2

(C, C, ...), (C, D, ...), ...

Payoffs (aka utilities)

In general, infinite sums.

For instance, if both players always cooperate, payoffs are infinite series: $(2, 2\delta, 2\delta^2, \ldots)$, and the final payoff is:

$$\begin{aligned} 2 + 2\delta + 2\delta^2 + \ldots &= 2(1 + \delta + \delta^2 + \ldots) \\ &= 2 \cdot \frac{1}{1-\delta} \end{aligned}$$

What does the discount factor δ stand for?

INTERPRETING THE DISCOUNT FACTOR

Patience

You’re more patient the less you mind waiting for something valuable, rather than receiving it immediately.

For a discount factor δ you value \$1, received t rounds from now, at $\$1 \cdot \delta^t$.

This is less than \$1, because $0 < \delta < 1$.

As δ gets closer to 1, the agent is more patient.

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As δ gets closer to 1, the agent is more patient.

Uncertainty about the future

You might prefer \$1 today to \$1 tomorrow because you're not sure tomorrow will even come.

δ can be the probability that there is a round $t + 1$, if round t has happened.

$\$1 \cdot \delta^t$ is then the expected payoff at round t.

ROBERT AUMANN

Consider, now, the following strategy, called *Grim Trigger*.

Start by cooperating. If the other player defects at some round t, switch to defecting forever, i.e., at every round $t' > t$.

Let’s look at a run of the game when one player plays Grim Trigger.

EXAMPLE RUNS WITH GRIM TRIGGER

Strategy of Player 1

Grim Trigger

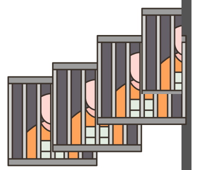
Strategy of Player 2

Start by cooperating; defect once at some random round $t > 1$

Sample run

	actions taken
Player 1	C, C, C, D, D, D, ...
Player 2	C, C, D, C, C, C, ...

Iterated Prisoner's Dilemma



infinitely iterated, with discount factor $0 < \delta < 1$

Two players play the regular Prisoner's Dilemma:

	C	D
C	2, 2	0, 3
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but an infinite number of times.

The final payoffs are the sum of the payoffs from each round, taking into account the discount factor δ.

EXAMPLE RUNS WITH GRIM TRIGGER

Strategy of Player 1

Grim Trigger

Strategy of Player 2

Start by cooperating; defect once at some random round $t > 1$

Sample run

revenge forever after

	actions taken
Player 1	C, C, C, D, D, D, …
Player 2	C, C, D, C, C, C, …

betrayal

Iterated Prisoner's Dilemma

infinitely iterated, with discount factor $0 < \delta < 1$

Two players play the regular Prisoner's Dilemma:

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

but an infinite number of times.

The final payoffs are the sum of the payoffs from each round, taking into account the discount factor δ.

1/2

EXAMPLE RUNS WITH GRIM TRIGGER

Strategy of Player 1
Grim Trigger

Strategy of Player 2
Start by cooperating; defect once at some random round $t > 1$

Sample run

revenge forever after

	actions taken	payoffs	total payoff
Player 1	C, C, C, D, D, D, ...	$2, 2\delta, 0\delta^2, 3\delta^3, 3\delta^4, 3\delta^5, \ldots$	the infinite sum
Player 2	C, C, D, C, C, C, ...	$2, 2\delta, 3\delta^2, 0\delta^3, 0\delta^4, 0\delta^5, \ldots$	the infinite sum

betrayal

Iterated Prisoner's Dilemma

infinitely iterated, with discount factor $0 < \delta < 1$

Two players play the regular Prisoner's Dilemma:

	C	D
C	2, 2	0, 3
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but an infinite number of times.

The final payoffs are the sum of the payoffs from each round, taking into account the discount factor δ.

And when both players use Grim Trigger?

EXAMPLE RUNS WITH GRIM TRIGGER

Strategy of Player 1

Grim Trigger

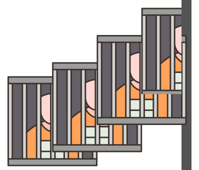
Strategy of Player 2

Grim Trigger

Sample run

	actions taken	payoffs	total payoff
Player 1	C, C, C, C, C, C, ...	$2, 2\delta, 2\delta^2, \ldots$	$2 \cdot ({}^1/{}_{1-\delta})$
Player 2	C, C, C, C, C, C, ...	$2, 2\delta, 2\delta^2, \ldots$	$2 \cdot ({}^1/{}_{1-\delta})$

Iterated Prisoner's Dilemma



infinitely iterated, with discount factor $0 < \delta < 1$

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but an infinite number of times.

The final payoffs are the sum of the payoffs from each round, taking into account the discount factor δ.

Does any agent have an incentive to deviate from Grim Trigger?

Player 2 deviates by always defecting

Strategy of Player 1

Grim Trigger

Strategy of Player 2

Deviate, starting at first round

Sample run

	actions taken	payoffs	total payoff
Player 1	C, D, D, D, D, D, ...	$0, \delta, \delta^2, \delta^3, \ldots$	$\delta/(1-\delta)$
Player 2	D, D, D, D, D, D, ...	$3, \delta, \delta^2, \delta^3, \ldots$	$2 + 1/(1-\delta)$

Iterated Prisoner's Dilemma

infinitely iterated, with discount factor $0 < \delta < 1$

Two players play the regular Prisoner's Dilemma:

	C	D
C	2, 2	0, 3
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but an infinite number of times.

The final payoffs are the sum of the payoffs from each round, taking into account the discount factor δ.

1/2

Player 2 deviates by always defecting

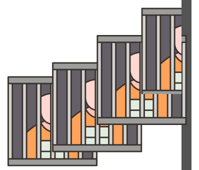
Strategy of Player 1
Grim Trigger

Strategy of Player 2
Deviate, starting at first round

Sample run

	actions taken	payoffs	total payoff
Player 1	C, D, D, D, D, D, ...	$0, \delta, \delta^2, \delta^3, \ldots$	$\delta/(1-\delta)$
Player 2	D, D, D, D, D, D, ...	$3, \delta, \delta^2, \delta^3, \ldots$	$2 + 1/(1-\delta)$

Iterated Prisoner's Dilemma



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Player 2 deviates by always defecting

Strategy of Player 1
Grim Trigger

Strategy of Player 2
Deviate, starting at first round

Sample run

	actions taken	payoffs	total payoff
Player 1	C, D, D, D, D, D, ...	$0, \delta, \delta^2, \delta^3, \ldots$	$\delta/(1-\delta)$
Player 2	D, D, D, D, D, D, ...	$3, \delta, \delta^2, \delta^3, \ldots$	$2 + 1/(1-\delta)$

Strategy of Player 1
Grim Trigger

Strategy of Player 2
Grim Trigger

Sample run

	actions taken	payoffs	total payoff
Player 1	C, C, C, C, C, C, ...	$2, 2\delta, 2\delta^2, \ldots$	$2 \cdot (1/1-\delta)$
Player 2	C, C, C, C, C, C, ...	$2, 2\delta, 2\delta^2, \ldots$	$2 \cdot (1/1-\delta)$

Iterated Prisoner's Dilemma

infinitely iterated, with discount factor $0 < \delta < 1$

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1/2

Player 2 deviates by always defecting

Strategy of Player 1
Grim Trigger

Strategy of Player 2
Deviate, starting at first round

Sample run

	actions taken	payoffs	total payoff
Player 1	C, D, D, D, D, D, ...	$0, \delta, \delta^2, \delta^3, \ldots$	$\delta/(1-\delta)$
Player 2	D, D, D, D, D, D, ...	$3, \delta, \delta^2, \delta^3, \ldots$	$2 + 1/(1-\delta)$

Strategy of Player 1
Grim Trigger

Strategy of Player 2
Grim Trigger

Sample run

	actions taken	payoffs	total payoff
Player 1	C, C, C, C, C, C, ...	$2, 2\delta, 2\delta^2, \ldots$	$2 \cdot (1/1-\delta)$
Player 2	C, C, C, C, C, C, ...	$2, 2\delta, 2\delta^2, \ldots$	$2 \cdot (1/1-\delta)$

Profitable?

Not a profitable deviation for Player 2 as long as:

$$2 + \frac{1}{1-\delta} \leq 2 \cdot \frac{1}{1-\delta},$$

which happens if and only if:

$$\delta \geq \frac{1}{2}$$

Iterated Prisoner's Dilemma

infinitely iterated, with discount factor $0 < \delta < 1$

Two players play the regular Prisoner's Dilemma:

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

but an infinite number of times.

The final payoffs are the sum of the payoffs from each round, taking into account the discount factor δ.

What if Player 2 defects later?

Player 2 deviates by defecting later

Strategy of Player 1

Grim Trigger

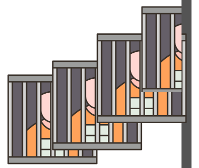
Strategy of Player 2

Deviate, starting at round $k > 1$

Sample run

	actions taken	payoffs	total payoff
Player 1	C, ..., C, C, D, D, ...	$2, 2\delta, \dots, 2\delta^{k-1}, 0\delta^k, \delta^{k+1}, \delta^{k+2}, \dots$	the infinite sum
Player 2	C, ..., C, D, D, D, ...	$2, 2\delta, \dots, 2\delta^{k-1}, 3\delta^k, \delta^{k+1}, \delta^{k+2}, \dots$	the infinite sum

Iterated Prisoner's Dilemma



infinitely iterated, with discount factor $0 < \delta < 1$

Two players play the regular Prisoner's Dilemma:

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

but an infinite number of times.

The final payoffs are the sum of the payoffs from each round, taking into account the discount factor δ.

Player 2 deviates by defecting later

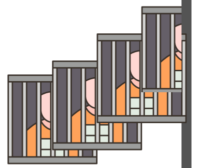
Strategy of Player 1
Grim Trigger

Strategy of Player 2
Deviate, starting at round $k > 1$

Sample run

	actions taken	payoffs	total payoff
Player 1	C, ..., C, C, D, D, ...	$2, 2\delta, \ldots, 2\delta^{k-1}, 0\delta^k, \delta^{k+1}, \delta^{k+2}, \ldots$	the infinite sum
Player 2	C, ..., C, D, D, D, ...	$2, 2\delta, \ldots, 2\delta^{k-1}, 3\delta^k, \delta^{k+1}, \delta^{k+2}, \ldots$	the infinite sum

Iterated Prisoner's Dilemma



infinitely iterated, with discount factor $0 < \delta < 1$

Two players play the regular Prisoner's Dilemma:

	C	D
C	$2,\ 2$	$0,\ 3$
D	$3,\ 0$	$1,\ 1$

but an infinite number of times.

The final payoffs are the sum of the payoffs from each round, taking into account the discount factor δ.

Player 2 deviates by defecting later

Strategy of Player 1
Grim Trigger

Strategy of Player 2
Deviate, starting at round $k > 1$

Sample run

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Player 2	C, ..., C, D, D, D, ...	$2, 2\delta, \ldots, 2\delta^{k-1}, 3\delta^{k}, \delta^{k+1}, \delta^{k+2}, \ldots$	the infinite sum

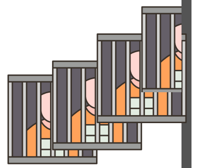
Strategy of Player 1
Grim Trigger

Strategy of Player 2
Grim Trigger

Sample run

	actions taken	payoffs	total payoff
Player 1	C, C, C, C, C, C, ...	$2, 2\delta, 2\delta^2, \ldots$	$2 \cdot (1/1-\delta)$
Player 2	C, C, C, C, C, C, ...	$2, 2\delta, 2\delta^2, \ldots$	$2 \cdot (1/1-\delta)$

Iterated Prisoner's Dilemma



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Strategy of Player 1
Grim Trigger

Strategy of Player 2
Grim Trigger

Sample run

	actions taken	payoffs	total payoff
Player 1	C, C, C, C, C, C, ...	$2, 2\delta, 2\delta^2, \ldots$	$2 \cdot (1/1-\delta)$
Player 2	C, C, C, C, C, C, ...	$2, 2\delta, 2\delta^2, \ldots$	$2 \cdot (1/1-\delta)$

Profitable?

Not a profitable deviation for Player 2 as long as:

$$
\begin{aligned}
2 + 2\delta + \cdots + 2\delta^{k-1} + 3\delta^k + \delta^{k+1} + \ldots \leq 2 + 2\delta + \cdots + 2\delta^{k-1} + 2\delta^k + 2\delta^{k+1} + \ldots \quad & \text{iff} \\
3\delta^k + \delta^{k+1} + \ldots \leq 2\delta^k + 2\delta^{k+1} + \ldots \quad & \text{iff} \\
3 + \delta + \delta^2 + \ldots \leq 2 + 2\delta + 2\delta^2 + \ldots \quad & \text{iff} \\
\delta \geq 1/2. \quad &
\end{aligned}
$$

Iterated Prisoner's Dilemma

infinitely iterated, with discount factor $0 < \delta < 1$

Two players play the regular Prisoner's Dilemma:

	C	D
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but an infinite number of times.

The final payoffs are the sum of the payoffs from each round, taking into account the discount factor δ.

Note that once Player 2 triggers Player 1 by defecting, Player 2 has no incentive to start cooperating again if all-defection is not profitable.

ROBERT AUMANN

We've just shown that if $\delta \geq 0.5$, no agent has an incentive to deviate.

In other words, both players playing Grim Trigger is a Nash equilibrium!

Finally, a positive result!

Infinite games (with sufficiently large discount factor) admit equilibria where players cooperate!

The moral?

If players send out a clear signal that they cannot be pushed around, it makes sense to cooperate.

ROBERT AUMANN

There’s many other ways of analyzing repeated games.

With or without discounting, with different ways of computing total payoffs, with different types of equilibria.

When these equilibria can be achieved is the subject of intense research.

Results here usually go under the name of *folk theorems*.

At the same time, Grim Trigger strategies are just one drop in the vast sea of possible strategies.

They are especially unforgiving, and do not match what we see in real life.

What else can we do?

ROBERT AXELROD

How about running a tournament...

Axelrod, R. (1984), *The Evolution of Cooperation*. Basic Books