TWEAKING DEMOCRACY

INNOVATIONS IN DEMOCRATIC DECISION MAKING

ONCE UPON A TIME IN AMERICA
THE HIGHS AND LOWS OF APPORTIONMENT

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NORTH AMERICA
1776

NORTH AMERICA
1776

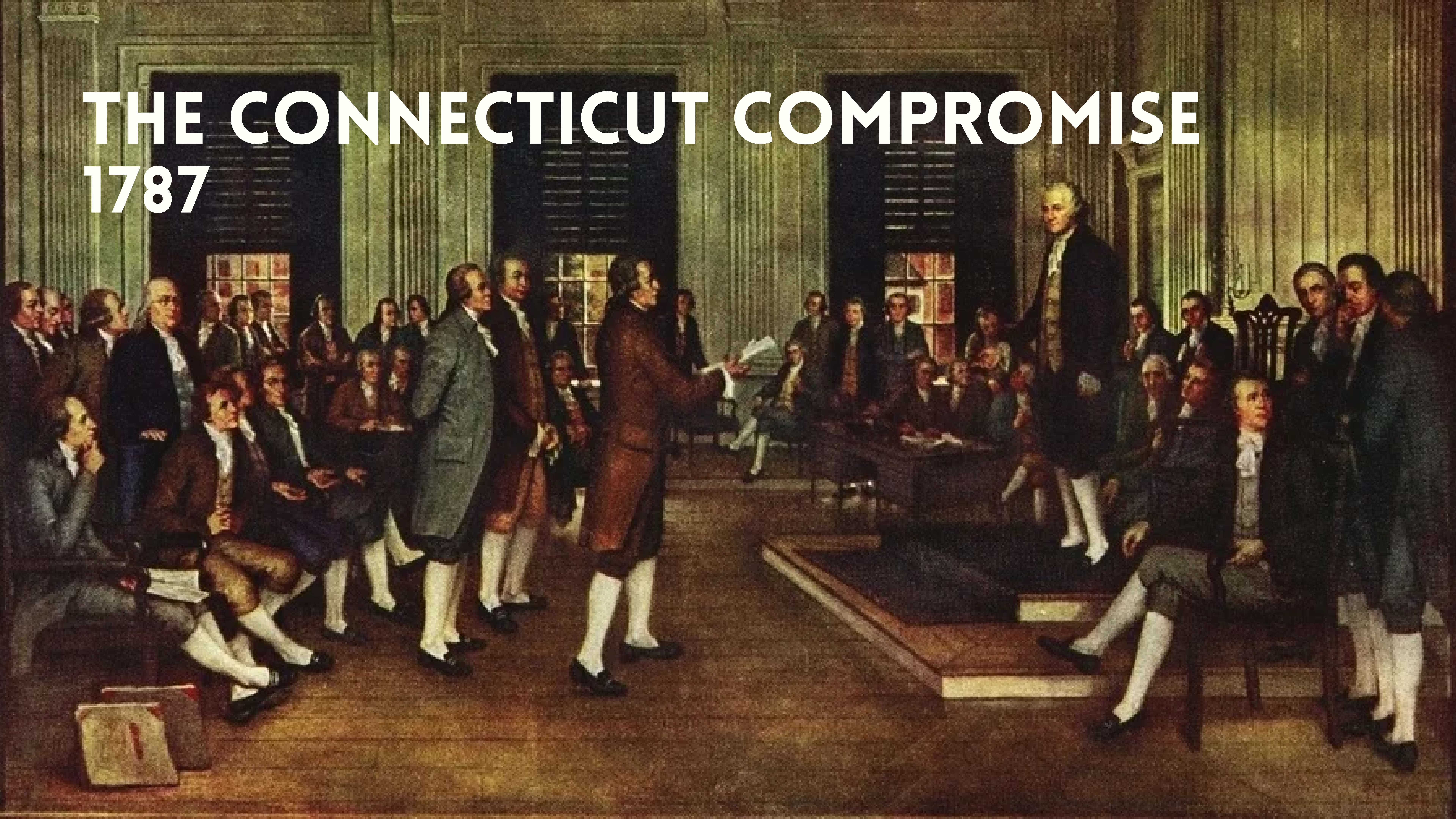
Thirteen colonies have had enough of being ruled by the British monarch.

And decide to splinter off into an independent state.

But the Founding Fathers discover that independence comes with its own set of problems...

How will the constituent states be represented at the national level?

THE CONNECTICUT COMPROMISE 1787



THE CONNECTICUT COMPROMISE
1787

States will be represented in the House of Representatives in a manner *proportional to their population.*

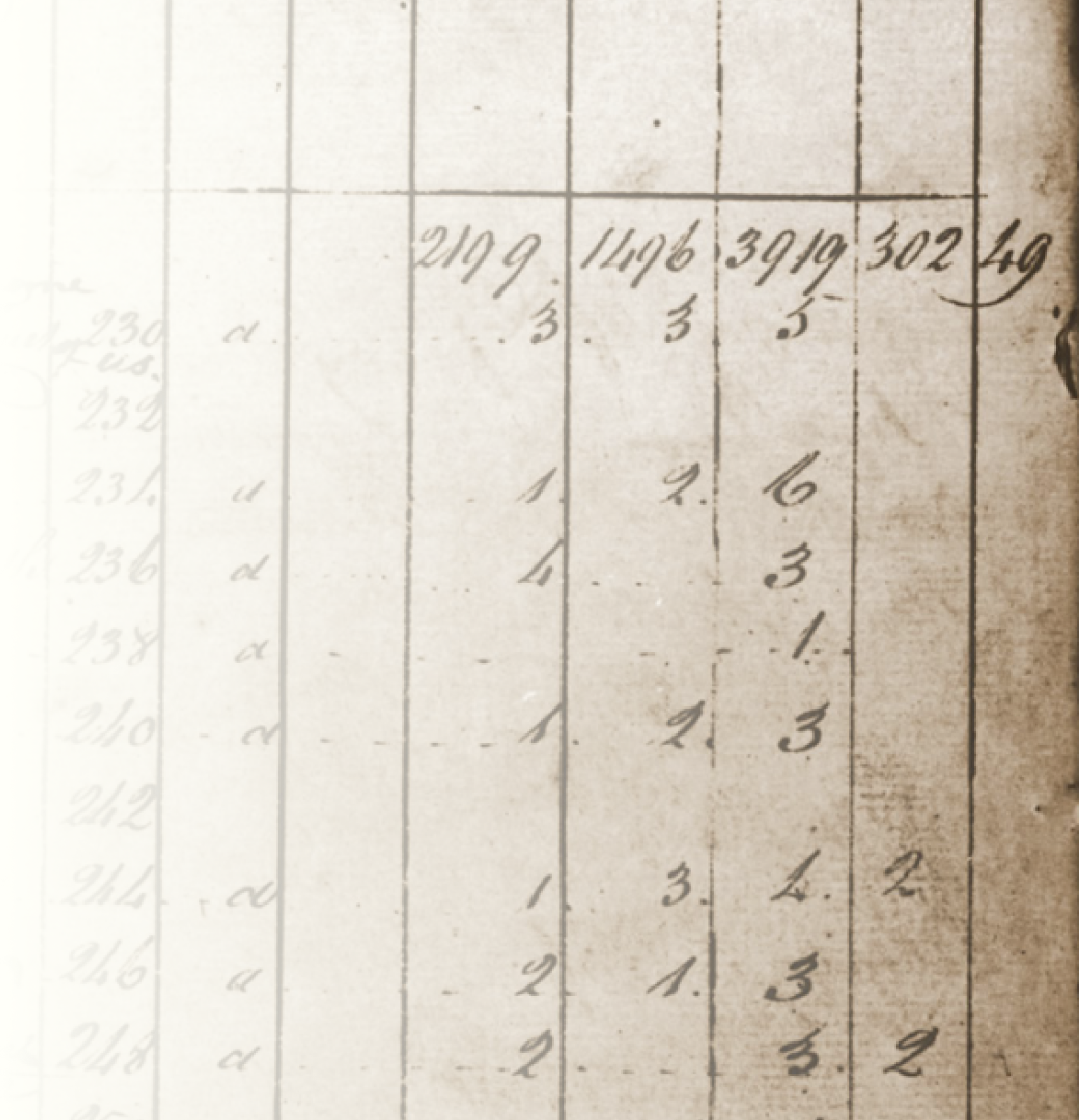
THE US CONSTITUTION
1789

> *Representatives [...] shall be apportioned among the several States [...] according to their respective Numbers.*
>
> *The Number of Representatives shall not exceed one for every thirty Thousand, but each State shall have at Least one Representative...*

US Constitution (1789), Article I, Section 2, Clause 3

THE FIRST US CENSUS
1790

Fifteen states.



THE FIRST US CENSUS
1790

Fifteen states.

state	population
Connecticut	236,841
Delaware	55,540
Georgia	70,835
Kentucky	68,705
Maryland	278,514
Massachusetts	475,327
New Hampshire	141,822
New Jersey	179,570
New York	331,589
North Carolina	353,523
Pennsylvania	432,879
Rhode Island	68,446
South Carolina	206,236
Vermont	85,533
Virginia	630,560
US (total)	3,615,920

THE FIRST US CENSUS
1790

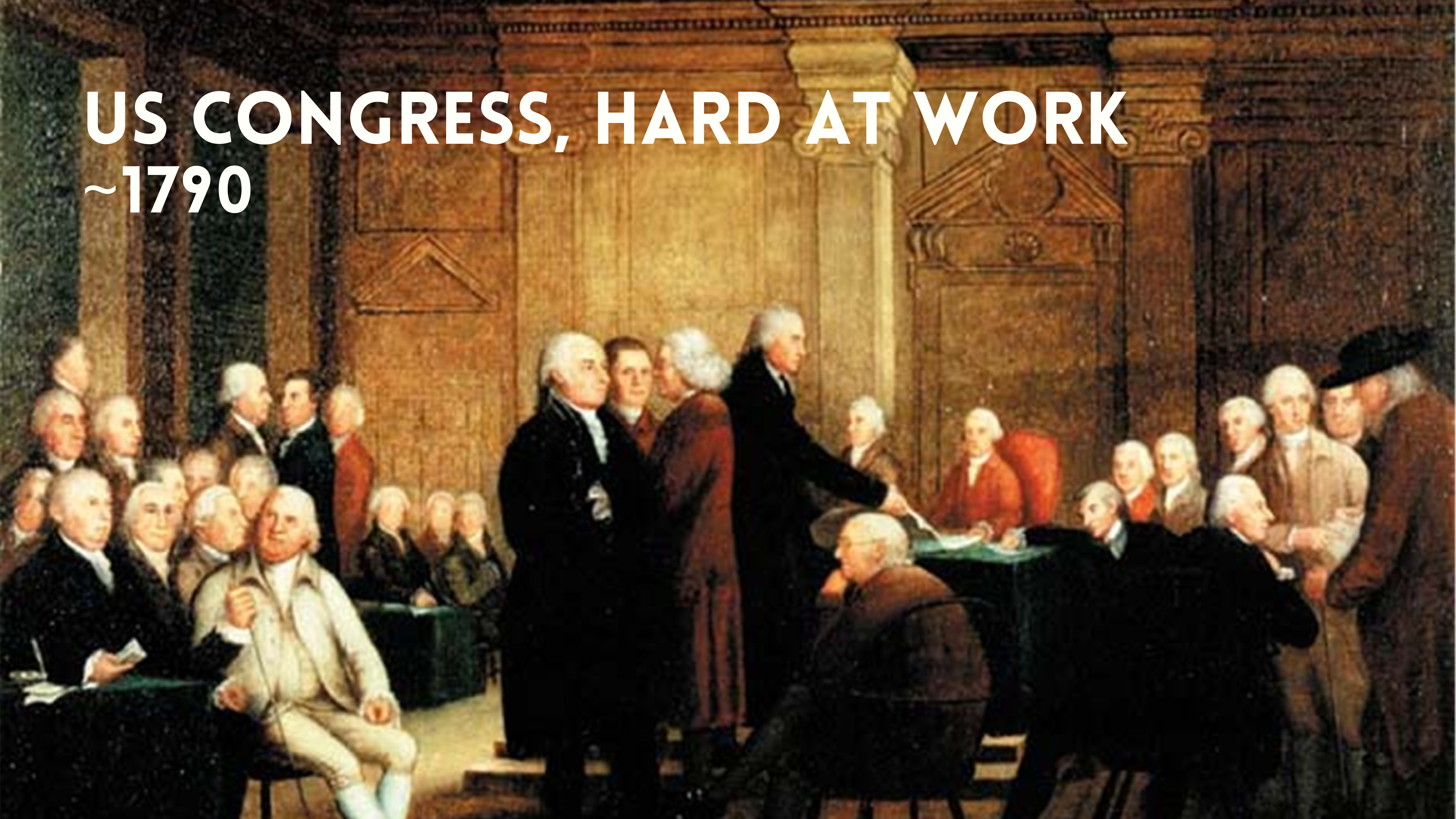
Fifteen states.

But the constitution does not specify how *exactly* to apportion representatives among them.

state	population
Connecticut	236,841
Delaware	55,540
Georgia	70,835
Kentucky	68,705
Maryland	278,514
Massachusetts	475,327
New Hampshire	141,822
New Jersey	179,570
New York	331,589
North Carolina	353,523
Pennsylvania	432,879
Rhode Island	68,446
South Carolina	206,236
Vermont	85,533
Virginia	630,560
US (total)	3,615,920

This makes things tricky…

US CONGRESS, HARD AT WORK ~1790



US CONGRESS, HARD AT WORK
~1790

Take one representative for every d persons, then let the number of representatives (house size) fall where it may.

$d = 30000$

QUOTAS

1

Choose a divisor *d*, the desired number of people per representative

For instance, $d = 30000$.

state	population
Connecticut	236,841
Delaware	55,540
Georgia	70,835
Kentucky	68,705
Maryland	278,514
Massachusetts	475,327
New Hampshire	141,822
New Jersey	179,570
New York	331,589
North Carolina	353,523
Pennsylvania	432,879
Rhode Island	68,446
South Carolina	206,236
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US (total)	3,615,920

QUOTAS

1 **Choose a divisor *d*, the desired number of people per representative**
For instance, $d = 30000$.

2 **Calculate quotas**
The *quota* of a state is its population divided by d.

$d = 30000$

state	population	population/d
Connecticut	236,841	7.895
Delaware	55,540	1.851
Georgia	70,835	2.361
Kentucky	68,705	2.29
Maryland	278,514	9.284
Massachusetts	475,327	15.844
New Hampshire	141,822	4.727
New Jersey	179,570	5.986
New York	331,589	11.053
North Carolina	353,523	11.784
Pennsylvania	432,879	14.429
Rhode Island	68,446	2.282
South Carolina	206,236	6.875
Vermont	85,533	2.851
Virginia	630,560	21.019
US (total)	3,615,920	120.531

QUOTAS

1 **Choose a divisor *d*, the desired number of people per representative**
For instance, $d = 30000$.

2 **Calculate quotas**
The *quota* of a state is its population divided by *d*.

3 **Assign seats as per quotas**
Ummm… what do we do about the fractions???

$d = 30000$

state	population	population/d	seats
Connecticut	236,841	7.895	?
Delaware	55,540	1.851	?
Georgia	70,835	2.361	?
Kentucky	68,705	2.29	?
Maryland	278,514	9.284	?
Massachusetts	475,327	15.844	?
New Hampshire	141,822	4.727	?
New Jersey	179,570	5.986	?
New York	331,589	11.053	?
North Carolina	353,523	11.784	?
Pennsylvania	432,879	14.429	?
Rhode Island	68,446	2.282	?
South Carolina	206,236	6.875	?
Vermont	85,533	2.851	?
Virginia	630,560	21.019	?
US (total)	3,615,920	120.531	?

US CONGRESS, HARD AT WORK
~1790

Let's just drop the fractions!

House Apportionment Bill of 1792

1 **Choose a divisor d, the desired number of people per representative**
For instance, $d = 30000$.

2 **Calculate each state's quota**
The *quota* of a state, i.e., its population divided by d, indicates the number of representatives the states deserves.

3 **Drop fractions and assign seats**
Leads to a house of size 112.

$d = 30000$

state	population	population/d	seats
Connecticut	236,841	7.895	7
Delaware	55,540	1.851	1
Georgia	70,835	2.361	2
Kentucky	68,705	2.29	2
Maryland	278,514	9.284	9
Massachusetts	475,327	15.844	15
New Hampshire	141,822	4.727	4
New Jersey	179,570	5.986	5
New York	331,589	11.053	11
North Carolina	353,523	11.784	11
Pennsylvania	432,879	14.429	14
Rhode Island	68,446	2.282	2
South Carolina	206,236	6.875	6
Vermont	85,533	2.851	2
Virginia	630,560	21.019	21
US (total)	3,615,920	120.531	112

Note that dropping of fractions tends to favor larger states.

Note that dropping of fractions tends to favor larger states.

We can see this by looking at the *representation ratio*, i.e., the number of people per representative a state gets from a particular assignment.

Large State Bias

Delaware vs Massachusetts

Dropping fractions hits different states differently.

Delaware ends up getting one seat for 55540 people, Massachusetts gets one seat for 31688 persons.

Every resident of Delaware has a 43% smaller share of representation in the House than a resident of Massachusetts.

House Apportionment Bill (1792)

$d = 30000$

state	population	population/d	seats	repr. ratio
Connecticut	236,841	7.895	7	33834.43
Delaware	55,540	1.851	1	55540
Georgia	70,835	2.361	2	35417.5
Kentucky	68,705	2.29	2	34352.5
Maryland	278,514	9.284	9	30946
Massachusetts	475,327	15.844	15	31688.47
New Hampshire	141,822	4.727	4	35455.5
New Jersey	179,570	5.986	5	35914
New York	331,589	11.053	11	30144.45
North Carolina	353,523	11.784	11	32138.45
Pennsylvania	432,879	14.429	14	30919.93
Rhode Island	68,446	2.282	2	34223
South Carolina	206,236	6.875	6	34372.67
Vermont	85,533	2.851	2	42766.5
Virginia	630,560	21.019	21	30026.67
US (total)	3,615,920	120.531	112	32285

The Senate disagreed with the House bill and proposed a different apportionment, by raising the divisor to 33000.

by the constitution, a divisor smaller than 30000 is not allowed

Senate Apportionment Bill of 1792

1 **Choose a divisor *d*, the desired number of people per representative**
$d = 33000$.

2 **Calculate each state's quota**
The *quota* of a state, i.e., its population divided by *d*, indicates the number of representatives the states deserves.

3 **Drop fractions and assign seats**
Leads to a house of size 105.

$d = 33000$

state	population	population/d	seats	repr. ratio
Connecticut	236,841	7.177	7	33834.43
Delaware	55,540	1.683	1	55540
Georgia	70,835	2.147	2	35417.5
Kentucky	68,705	2.082	2	34352.5
Maryland	278,514	8.44	8	34814.25
Massachusetts	475,327	14.404	14	33951.93
New Hampshire	141,822	4.298	4	35455.5
New Jersey	179,570	5.442	5	35914
New York	331,589	10.048	10	33158.9
North Carolina	353,523	10.713	10	35352.3
Pennsylvania	432,879	13.118	13	33298.38
Rhode Island	68,446	2.074	2	34223
South Carolina	206,236	6.25	6	34372.67
Vermont	85,533	2.592	2	42766.5
Virginia	630,560	19.108	19	33187.37
US (total)	3,615,920	109.573	105	34437.333

All the wrangling over divisors came across as silly.

Edmund Ranolph

1753 - 1813

Founding father of the United States, attorney, seventh governor of Virginia.

Thought the hunt for divisors was silly.

"Sir, it gave me pain to find these worthy members calculating and coldly applying rules of arithmetic to a subject beyond the power of numbers to express the degree of its importance to their fellow citizens."

At the same time, every state fought fiercely for every seat.

The dispute had added weight given the growing divide between North and South.

Enter Hamilton.

Alexander Hamilton

1757 - 1804

Founding father of the United States.

Played a key role in securing America's independence, and pushing through the Constitution.

Died in a duel with political rival Aaron Burr.

These days, famous mostly for starring in musicals.

ALEXANDER HAMILTON

The whole number of Representatives being first fixed, they shall be apportioned to any state according to its census…

This number should probs be 120, approx. corresponding to the total population of the US divided by 30000.

Let us call this the true, or standard, quota.

…the Rule of Three will show what part of the representation any State shall have…

In other words, the total number of seats to be distributed should be fixed in advance.

The share of each state is then calculated in proportion to its percentage of the population.

Glossary of Terms

states	$N = \{1, \ldots, n\}$
population of state i	p_i
total population	$p = p_1 + \ldots + p_n$
number of seats to be allocated	k
seats allocated to state i	k_i
divisor	d
quota of state i, for divisor d	$\hat{q}_i = {}^{p_i}/_{d}$
standard (true) quota of state i	$q_i = {}^{p_i}/_{p} \cdot k$
upper quota of state i	$\lceil q_i \rceil$, i.e., q_i rounded up to the nearest integer
lower quota of state i	$\lfloor q_i \rfloor$, i.e., q_i rounded down to the nearest integer

corresponds to a divisor of $\frac{p}{k}$

ALEXANDER HAMILTON

Fix the number k of seats to be allocated.

Start by giving each state its lower standard quota.

If there are seats that remain to be allocated, look at the residue of each state:

$$r_i = q_i - \lfloor q_i \rfloor$$

Distribute the remaining seats (one each) to the states with the largest residues.

Hamilton's Method

1

Every state gets its lower standard quota

There are 9 remaining seats to be allocated.

$d = 30132.67$

state	population	population/d	seats	repr. ratio
Connecticut	236,841	7.86	7	33834.43
Delaware	55,540	1.843	1	55540
Georgia	70,835	2.351	2	35417.5
Kentucky	68,705	2.28	2	34352.5
Maryland	278,514	9.243	9	30946
Massachusetts	475,327	15.774	15	31688.47
New Hampshire	141,822	4.707	4	35455.5
New Jersey	179,570	5.959	5	35914
New York	331,589	11.004	11	30144.45
North Carolina	353,523	11.732	11	32138.45
Pennsylvania	432,879	14.366	14	30919.93
Rhode Island	68,446	2.271	2	34223
South Carolina	206,236	6.844	6	34372.67
Vermont	85,533	2.839	2	42766.5
Virginia	630,560	20.926	20	31528
US (total)	3,615,920	120	111	32575.856

Hamilton's Method

1 **Every state gets its lower standard quota**

There are 9 remaining seats to be allocated.

2 **Order states by remainder**

Connecticut, Delaware, Massachusetts, New Hampshire, New Jersey, North Carolina, South Carolina, Vermont and Virginia are the 9 states with the highest remainders.

$d = 30132.67$

state	population	population/d	seats	repr. ratio
Connecticut	236,841	7.86	7	33834.43
Delaware	55,540	1.843	1	55540
Georgia	70,835	2.351	2	35417.5
Kentucky	68,705	2.28	2	34352.5
Maryland	278,514	9.243	9	30946
Massachusetts	475,327	15.774	15	31688.47
New Hampshire	141,822	4.707	4	35455.5
New Jersey	179,570	5.959	5	35914
New York	331,589	11.004	11	30144.45
North Carolina	353,523	11.732	11	32138.45
Pennsylvania	432,879	14.366	14	30919.93
Rhode Island	68,446	2.271	2	34223
South Carolina	206,236	6.844	6	34372.67
Vermont	85,533	2.839	2	42766.5
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US (total)	3,615,920	120	111	32575.856

Hamilton's Method

1

Every state gets its lower standard quota

There are 9 remaining seats to be allocated.

2

Order states by remainder

Connecticut, Delaware, Massachusetts, New Hampshire, New Jersey, North Carolina, South Carolina, Vermont and Virginia are the 9 states with the highest remainders.

3

Allocate the remaining seats

These states get an extra seat each.

$d = 30132.67$

state	population	population/d	seats	repr. ratio
Connecticut	236,841	7.86	8	29605.13
Delaware	55,540	1.843	2	27770
Georgia	70,835	2.351	2	35417.5
Kentucky	68,705	2.28	2	34352.5
Maryland	278,514	9.243	9	30946
Massachusetts	475,327	15.774	16	29707.94
New Hampshire	141,822	4.707	5	28364.4
New Jersey	179,570	5.959	6	29928.33
New York	331,589	11.004	11	30144.45
North Carolina	353,523	11.732	12	29460.25
Pennsylvania	432,879	14.366	14	30919.93
Rhode Island	68,446	2.271	2	34223
South Carolina	206,236	6.844	7	29462.29
Vermont	85,533	2.839	3	28511
Virginia	630,560	20.926	21	30026.67
US (total)	3,615,920	120	120	30132.667

A compromise bill with this exact apportionment was passed by narrow majorities on March 26, 1792.

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All that remained was for President George Washington to sign it.

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All that remained was for President George Washington to sign it.

He had until April 5 to make a decision…

Enter Washington.

George Washington

1732 - 1799

Founding father of the United States,
general, first president.

Defeated the British, ensuring the
independence of the US.

Refused the title of king, stayed on for
two spells as president.

Father of the nation.

GEORGE WASHINGTON

So I *guess* I *should sign the compromise bill?*

GEORGE WASHINGTON

So I *guess* I *should sign the compromise bill?*

ALEXANDER HAMILTON

Oh for sure!

It results from a logical method, that works for any situation…

GEORGE WASHINGTON

So I guess I should sign the compromise bill?

ALEXANDER HAMILTON

Oh for sure!

It results from a logical method, that works for any situation...

THOMAS JEFFERSON

Not so fast!

Enter Jefferson.

Thomas Jefferson

1743 - 1826

Founding father, primary author of the Declaration of Independence, secretary of state under George Washington.

Went on to become the third president of the US.

During his tenure the US would double in size.

Lives on as the face on the nickel, as a member of the Mount Rushmore four, and as a champion of freedom and democracy (who also owned slaves).

THOMAS JEFFERSON

Hamilton's doctrine of fractions is difficult and unobvious.

EDMUND RANDOLPH

THOMAS JEFFERSON

Hamilton's doctrine of fractions is difficult and unobvious.

EDMUND RANDOLPH

I *agree!*

THOMAS JEFFERSON

Hamilton's doctrine of fractions is difficult and unobvious.

EDMUND RANDOLPH

I *agree!*

In fact, by Hamilton's method, all states whose delegation is rounded up get more than one representative for 30000 residents.

For instance, New Hampshire would get one representative per 28364 citizens.

New Hampshire	141,822	4.707	5	28364.4

This is unconstitutional!

Interestingly, both Jefferson and Randolph hailed from Virginia, a state that would not benefit from rounding up.

Interestingly, both Jefferson and Randolph hailed from Virginia, a state that would not benefit from rounding up.

But surely that was a coincidence…

GEORGE WASHINGTON

What a nuisance!

This apportionment issue is pitching Northern *states versus* Southern *states.*

But I *do not want to take a side.*

April 5 arrives and Washington is yet to make a decision…

GEORGE WASHINGTON

Jefferson! In my office! Now!

THOMAS JEFFERSON

GEORGE WASHINGTON

Jefferson! In my office! Now!

THOMAS JEFFERSON

But I have not even had breakfast yet...

GEORGE WASHINGTON

Jefferson! In my office! Now!

THOMAS JEFFERSON

But I *have not even had breakfast yet...*

GEORGE WASHINGTON

What shall we do?

GEORGE WASHINGTON

Jefferson! In my office! Now!

THOMAS JEFFERSON

But I have not even had breakfast yet…

GEORGE WASHINGTON

What shall we do?

THOMAS JEFFERSON

You should negative the bill…

Washington vetoes the bill (!).

GEORGE WASHINGTON

What now?

THOMAS JEFFERSON

GEORGE WASHINGTON
What now?

THOMAS JEFFERSON
Here's *what* I *propose.*

GEORGE WASHINGTON

What now?

THOMAS JEFFERSON

Here's what I propose.

Start with the desired number of seats k.

Find a divisor d such that:

$$\left\lfloor \frac{p_1}{d} \right\rfloor + \ldots + \left\lfloor \frac{p_n}{d} \right\rfloor = k$$

State i gets $\lfloor p_i/d \rfloor$ seats.

Jefferson's Method

1

Choose the house size

Say we want $k = 120$ seats.

state	population	population/d	seats	repr. ratio
Connecticut	236,841			
Delaware	55,540			
Georgia	70,835			
Kentucky	68,705			
Maryland	278,514			
Massachusetts	475,327			
New Hampshire	141,822			
New Jersey	179,570			
New York	331,589			
North Carolina	353,523			
Pennsylvania	432,879			
Rhode Island	68,446			
South Carolina	206,236			
Vermont	85,533			
Virginia	630,560			
US (total)	3,615,920		120	

Jefferson's Method

1

Choose the house size

Say we want $k = 120$ seats.

2

Find the right divisor

30000 doesn't work, use $d = 28500$.*

*For this case any divisor between 28356 and 28511 works.

$d = 28500$

state	population	population/d	seats	repr. ratio
Connecticut	236,841	8.31		
Delaware	55,540	1.949		
Georgia	70,835	2.485		
Kentucky	68,705	2.411		
Maryland	278,514	9.772		
Massachusetts	475,327	16.678		
New Hampshire	141,822	4.976		
New Jersey	179,570	6.301		
New York	331,589	11.635		
North Carolina	353,523	12.404		
Pennsylvania	432,879	15.189		
Rhode Island	68,446	2.402		
South Carolina	206,236	7.236		
Vermont	85,533	3.001		
Virginia	630,560	22.125		
US (total)	3,615,920	126.874	120	

Jefferson's Method

1 **Choose the house size**

Say we want $k = 120$ seats.

2 **Find the right divisor**

30000 doesn't work, use $d = 28500$.*

*For this case any divisor between 28356 and 28511 works.

3 **Assign seats**

Round down.

$d = 28500$

state	population	population/d	seats	repr. ratio
Connecticut	236,841	8.31	8	29605.13
Delaware	55,540	1.949	1	55540
Georgia	70,835	2.485	2	35417.5
Kentucky	68,705	2.411	2	34352.5
Maryland	278,514	9.772	9	30946
Massachusetts	475,327	16.678	16	29707.94
New Hampshire	141,822	4.976	4	35455.5
New Jersey	179,570	6.301	6	29928.33
New York	331,589	11.635	11	30144.45
North Carolina	353,523	12.404	12	29460.25
Pennsylvania	432,879	15.189	15	28858.6
Rhode Island	68,446	2.402	2	34223
South Carolina	206,236	7.236	7	29462.29
Vermont	85,533	3.001	3	28511
Virginia	630,560	22.125	22	28661.82
US (total)	3,615,920	126.874	120	30132.667

Jefferson's Method

1 **Choose the house size**

Say we want $k = 120$ seats.

2 **Find the right divisor**

30000 doesn't work, use $d = 28500$.*

*For this case any divisor between 28356 and 28511 works.

3 **Assign seats**

Round down.

$d = 28500$

state	population	population/d	seats	repr. ratio
Connecticut	236,841	8.31	8	29605.13
Delaware	55,540	1.949	1	55540
Georgia	70,835	2.485	2	35417.5
Kentucky	68,705	2.411	2	34352.5
Maryland	278,514	9.772	9	30946
Massachusetts	475,327	16.678	16	29707.94
New Hampshire	141,822	4.976	4	35455.5
New Jersey	179,570	6.301	6	29928.33
New York	331,589	11.635	11	30144.45
North Carolina	353,523	12.404	12	29460.25
Pennsylvania	432,879	15.189	15	28858.6
Rhode Island	68,446	2.402	2	34223
South Carolina	206,236	7.236	7	29462.29
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US (total)	3,615,920	126.874	120	30132.667

GEORGE WASHINGTON

No bueno! A representation ratio smaller than 30000 landed us in this mess in the first place!

THOMAS JEFFERSON

GEORGE WASHINGTON

No bueno! A representation ratio smaller than 30000 landed us in this mess in the first place!

THOMAS JEFFERSON

My bad!

To get better representation ratios we'll need to raise the divisor.

A bigger divisor leads to a smaller house though…

Two days later a new bill was proposed, using Jefferson's method with a divisor of 33000 and a house size of 105.

Two days later a new bill was proposed, using Jefferson's method with a divisor of 33000 and a house size of 105.

The Senate voted for it on the same day, and Washington signed the bill into law on April 14, 1792.

Jefferson had triumphed.

His method was used until the 1830s.

Jefferson had triumphed.

His method was used until the 1830s.

Until some states noticed something fishy…

Jefferson's method favors large states.

Large State Bias of Jefferson's Method

We want to distribute 100 seats among a population of 10,000,000. Thus, ideally, around 100,000 people per representative.

$d = 100{,}000$

state	population	population/d	seats	repr. ratio
New York	2,620,000	26.2	26	100769.23
Delaware	168,000	1.68	1	168000
...	...	...	...	...
US (total)	10,000,000	100	99	101010.101

Large State Bias of Jefferson's Method

We want to distribute 100 seats among a population of 10,000,000. Thus, ideally, around 100,000 people per representative.

But the divisor $d = 100{,}000$ does not deliver enough seats.

$d = 100{,}000$

state	population	population/d	seats	repr. ratio
New York	2,620,000	26.2	26	100769.23
Delaware	168,000	1.68	1	168000
...	...	...	...	...
US (total)	10,000,000	100	99	101010.101

Large State Bias of Jefferson's Method

We want to distribute 100 seats among a population of 10,000,000. Thus, ideally, around 100,000 people per representative.

But the divisor $d = 100{,}000$ does not deliver enough seats.

Decreasing the divisor to $d' = 97{,}000$ does the trick, but the additional seat goes to the larger state (New York).

Larger states arrive 'earlier' at the additional seat.

$d = 100{,}000$

state	population	population/d	seats	repr. ratio
New York	2,620,000	26.2	26	100769.23
Delaware	168,000	1.68	1	168000
...	...	...	...	...
US (total)	10,000,000	100	99	101010.101

$d' = 97{,}000$

state	population	population/d'	seats	repr. ratio
New York	2,620,000	27.01	27	97037.04
Delaware	168,000	1.732	1	168000
...	...	...	...	...
US (total)	10,000,000	103.093	100	100000

Jefferson’s method disenfranchises voters in the left out fractions of small states.

Enter Lowndes.

William Jones Lowndes

1782 - 1822

Congressman from South Carolina.

Involved in negotiations around the Missouri compromise.

Proposed a new apportionment method.

WILLIAM J. LOWNDES

Start with the desired number of seats k.

Calculate the standard quota of each seat and round down, like with Hamilton's method.

Divide the quotas by the initial number of seats given.

Assign remaining seats in order of this new quantity.

Lowndes' Method

1

Every state gets its lower standard quota

There are 13 out of the desired 213 seats left to be allocated.

state	population	standard quota	initial seats
Pennsylvania	1,049,313	24.917	24
Illinois	54,843	1.302	1
...	...	...	...
Total			200

Lowndes’ Method

1

Every state gets its lower standard quota

There are 13 out of the desired 213 seats left to be allocated.

2

Order states by priority number, calculated as their standard quota divided by the number of inisial seats

Note that Illinois has a higher priority number than Pennsylvania.

state	population	standard quota	initial seats	st q/i seats
Pennsylvania	1,049,313	24.917	24	1.04
Illinois	54,843	1.302	1	1.30
...	...	...	...	...
Total			200	

Lowndes' Method

1

Every state gets its lower standard quota

There are 13 out of the desired 213 seats left to be allocated.

2

Order states by priority number, calculated as their standard quota divided by the number of inisial seats

Note that Illinois has a higher priority number than Pennsylvania.

3

Allocate the remaining seats

Unlike with Hamilton's method, Illinois gets an extra seat before Pennsylvania.

state	population	standard quota	initial seats	st q/i seats	final seats
Pennsylvania	1,049,313	24.917	24	1.04	24
Illinois	54,843	1.302	1	1.30	2
...	...	...	...	...	...
Total			200		213

In 1820, Lowndes' method would have given all the extra seats to the smallest states.

It was promptly rejected by Congress.

Enter Adams.

John Adams

1735 - 1826

Founding father, and second president of the US.

While president, he waged an unofficial naval war with France.

According to Benjamin Franklin, "He means well for his country, is always an honest man, often a wise one, but sometimes, and in some things, absolutely out of his senses."

JOHN ADAMS

Start with the desired number of seats k.

Find a divisor d such that:

$$\left\lceil \frac{p_1}{d} \right\rceil + \ldots + \left\lceil \frac{p_n}{d} \right\rceil = k$$

State i gets $\left\lceil p_i / d \right\rceil$ seats.

Unsurprisingly, Adams' method favors small states.

Small State Bias of Adams' Method

We want to distribute 100 seats among a population of 10,000,000. This means around 100,000 people per representative.

The divisor $d = 100{,}000$ does not deliver enough seats.

Here we need to *increase* the divisor to $d' = 104{,}000$ to get the desired number of seats.

But now the small states get an advantage.

$d = 100{,}000$

state	population	population/d	seats	repr. ratio
New York	2,668,000	26.68	27	98814.81
Delaware	120,000	1.2	2	60000
...	...	...	...	...
US (total)	10,000,000	100	101	99009.901

$d' = 104{,}000$

state	population	population/d'	seats	repr. ratio
New York	2,668,000	25.654	26	102615.38
Delaware	120,000	1.154	2	60000
...	...	...	...	...
US (total)	10,000,000	96.154	100	100000

Adams' method was considered by Congress, but never enacted.

Adams' method was considered by Congress, but never enacted.

The larger states, having the upper hand, would have none of it.

JOHN ADAMS

I *hung my harp upon my willows, and gave up.*

Enter Webster.

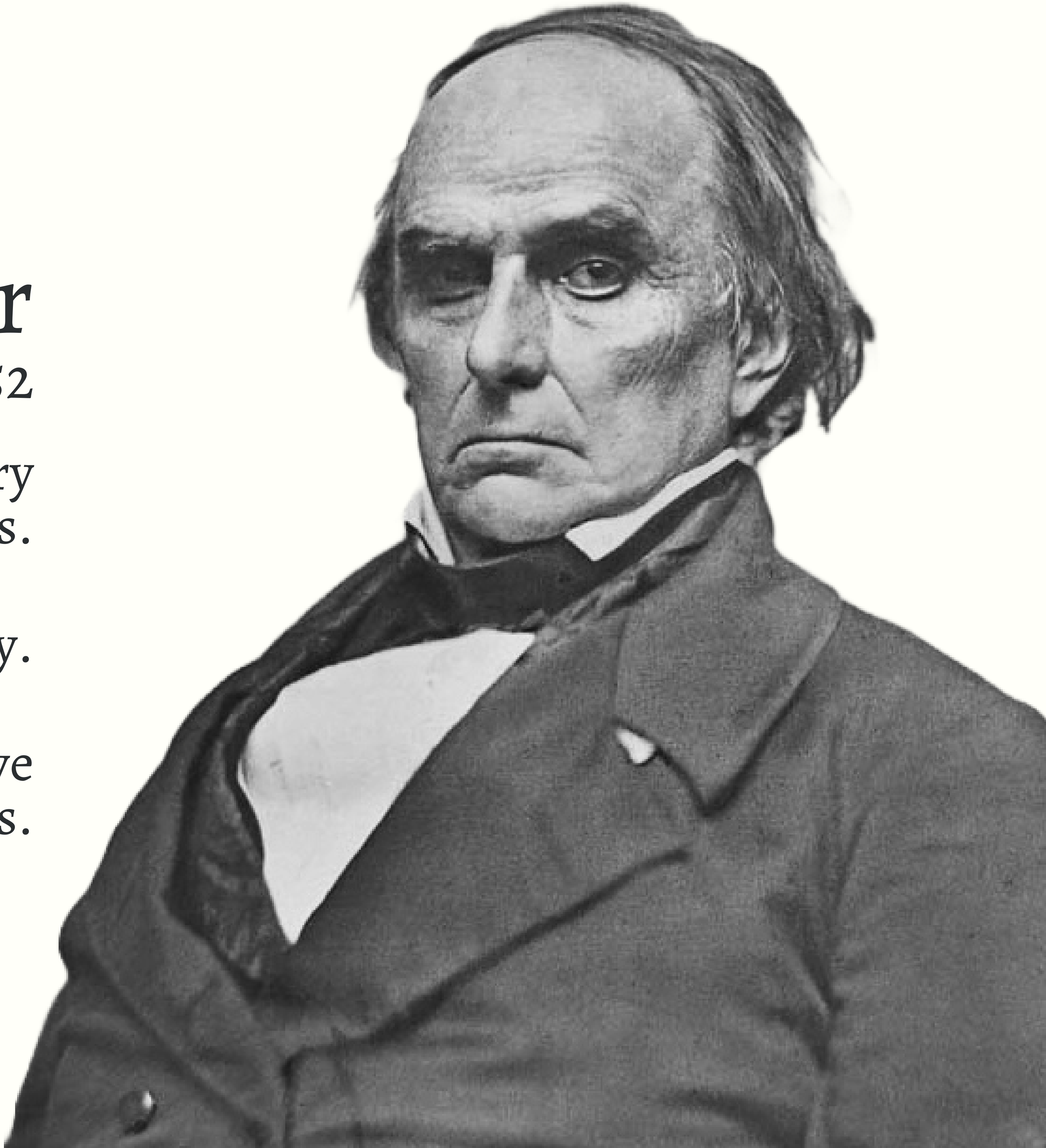
Daniel Webster

1782 - 1852

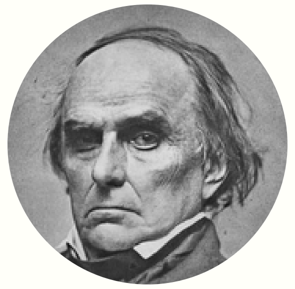
Lawyer, congressman, and US secretary of state under three presidents.

Famous for his oratory.

His speeches were reported to move even the most stone-hearted to tears.



DANIEL WEBSTER



Start with the desired number of seats k.

Find a divisor d such that:

$$\left[\frac{p_1}{d}\right] + \ldots + \left[\frac{p_n}{d}\right] = k$$

State i gets $[p_i/d]$ seats.

nearest integer

Webster's Method Is Impartial

We want to distribute 33 seats among a population of 330,000. This means 10,000 people per representative.

The divisor $d = 10{,}000$, together with Webster's method, delivers the right number of seats.

Rounding to the nearest integer sometimes favors the smaller state, sometimes the larger state.

$d = 10{,}000$

state	population	population/d	seats	repr. ratio
Colorado	304,000	30.4	30	10133.33
Nebraska	26,000	2.6	3	8666.67
US (total)	330,000	33	33	10000

$d = 10{,}000$

state	population	population/d	seats	repr. ratio
Oregon	296,000	29.6	30	9866.67
Arkansas	34,000	3.4	3	11333.33
US (total)	330,000	33	33	10000

Webster's method was adopted in 1842.

Webster's method was adopted in 1842.

Not ten years passed until it was challenged.

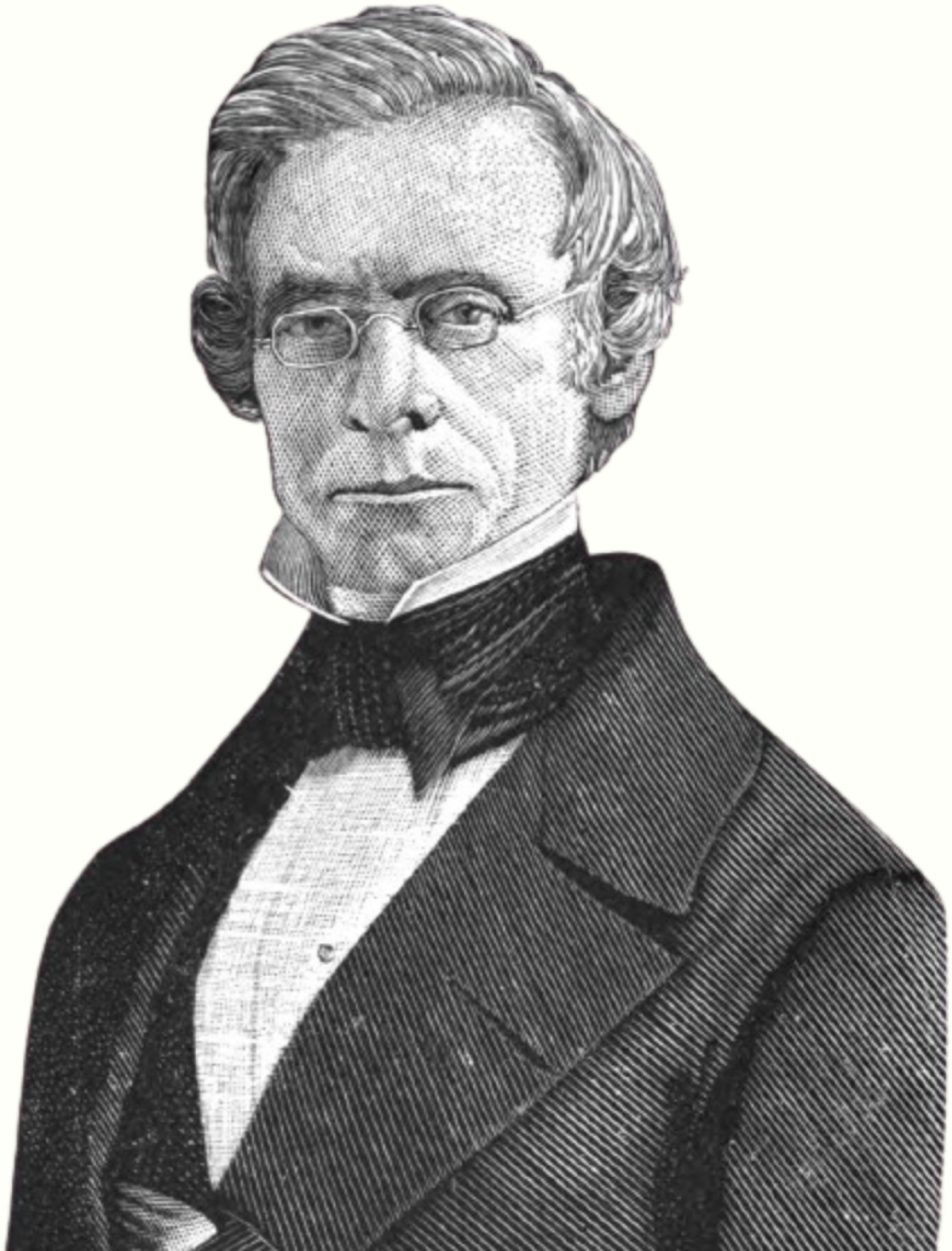
Enter Vinton.

Samuel Finley Vinton

1792 - 1862

Member of the House of Representatives, hailing from Ohio.

Helped create the US Department of the Interior.



SAMUEL F. VINTON

Fix the number k of seats to be allocated.

Start by giving each state its lower standard quota.

If there are seats that remain to be allocated, look at the residue of each state:

$$r_i = q_i - \lfloor q_i \rfloor$$

Distribute the remaining seats (one each) to the states with the largest residues.

Vinton’s method was, of course, identical to the method proposed by Hamilton and which had been vetoed by Washington in 1792.

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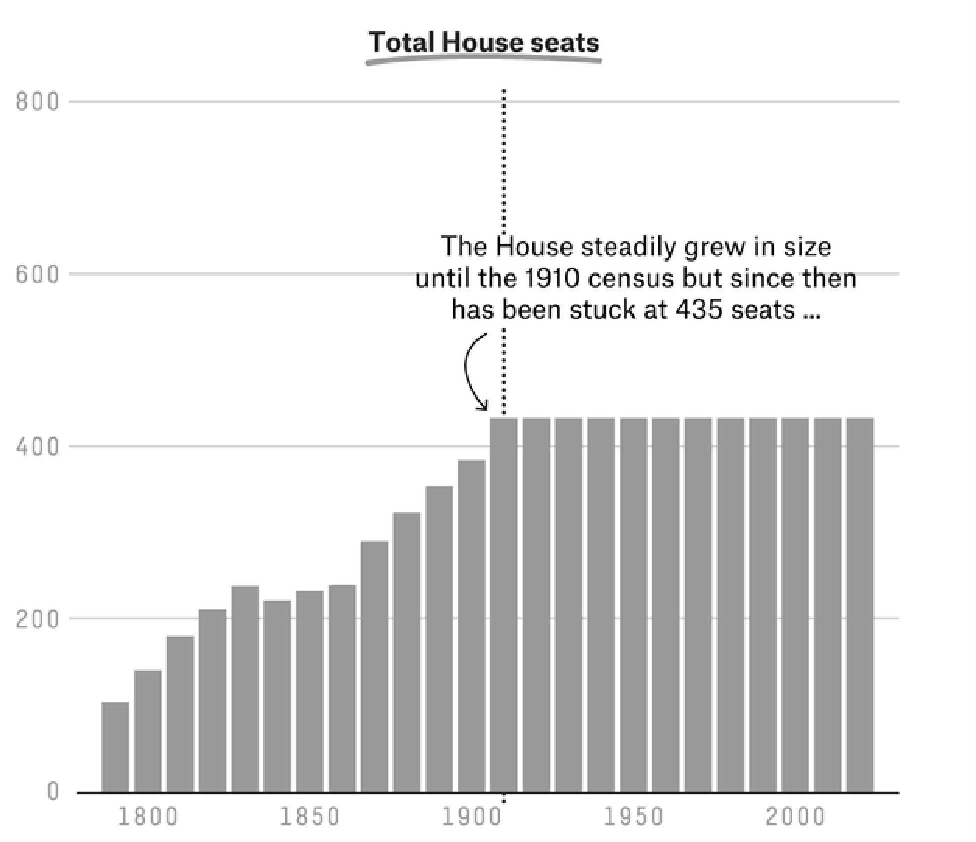
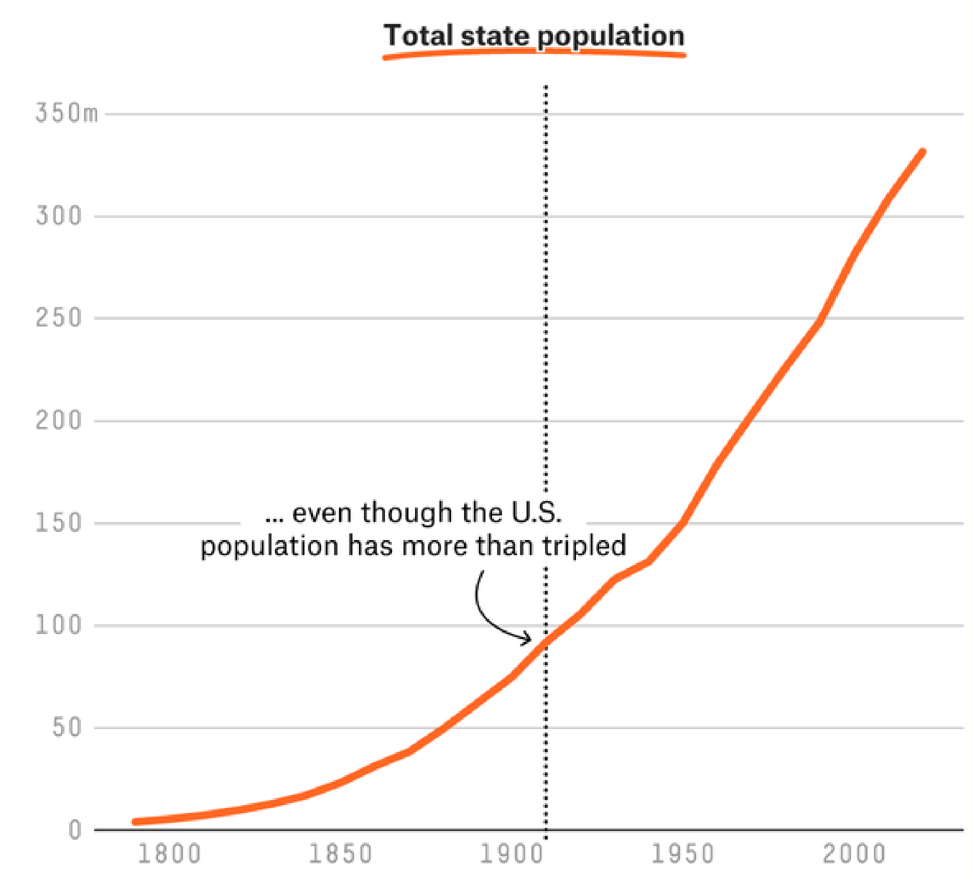
Congress adopted it in 1850.

Meanwhile, the population of the US keeps growing, with the House struggling to keep up.





Skelley, G. (2021). How The House Got Stuck At 435 Seats. *FiveThirtyEight*.

After the 1880 census, the House was expected to grow again.

After the 1880 census, the House was expected to grow again.

But when the seats were computed, something extraordinary happened…

The Alabama Paradox

We start with $k = 299$ seats, to be distributed among a population of ~50 mil.

With the (standard) divisor $d = 165{,}120$, the Hamilton-Vinton method gives Alabama 8 seats.

$d = 165{,}120$

state	population	population/d	seats
Alabama	1,262,505	7.646	8
Texas	1,591,749	9.64	9
Illinois	3,077,871	18.64	18
...	...	...	...
US (total)	49,713,370	301.074	299

The Alabama Paradox

We start with $k = 299$ seats, to be distributed among a population of ~50 mil.

With the (standard) divisor $d = 165{,}120$, the Hamilton-Vinton method gives Alabama 8 seats.

Increasing the House size to $k + 1 = 300$ (and recalculating the divisor to $d' = 164{,}580$) results in Alabama *losing* a seat!

$d = 165{,}120$

state	population	population/d	seats
Alabama	1,262,505	7.646	8
Texas	1,591,749	9.64	9
Illinois	3,077,871	18.64	18
...	...	...	...
US (total)	49,713,370	301.074	299

$d' = 164{,}580$

state	population	population/d'	seats
Alabama	1,262,505	7.671	7
Texas	1,591,749	9.672	10
Illinois	3,077,871	18.701	19
...	...	...	...
US (total)	49,713,370	302.062	300

Members of Congress were outraged.

Members of Congress were outraged.

The compromise solution was to enlarge the House to 325 seats, on which Webster's and Hamilton's methods agreed.

Soon enough, another problem emerged.

The Population Paradox

In 1900 the size of the house had risen to $k = 386$ seats, to be distributed among a population of ~74.5 mil.

The Hamilton-Vinton method gives Virginia 8 seats.

$d \sim 193{,}164$

state	population	population/d	seats
Virginia	1,854,184	9.599	10
Maine	694,466	3.595	3
...	...	...	...
US (total)	74,562,608	386.006	386

The Population Paradox

In 1900 the size of the house had risen to $k = 386$ seats, to be distributed among a population of ~74.5 mil.

The Hamilton-Vinton method gives Virginia 8 seats.

A year later, Virginia's population grew by 1.06%, while Maine's grew by 0.7%.

But the extra seat goes to Maine!

$d \sim 193{,}164$

state	population	population/d	seats
Virginia	1,854,184	9.599	10
Maine	694,466	3.595	3
...	...	...	...
US (total)	74,562,608	386.006	386

$d \sim 197{,}071$

state	population	population/d	seats
Virginia	1,873,951	9.509	9
Maine	699,114	3.548	4
...	...	...	...
US (total)	76,069,522	386	386

And another problem.

The New State Paradox

In 1907, Oklahoma joined the union.

At around 1 million people, Oklahoma deserved five seats in the House.

Congress then added five seats, and used Hamilton's method to recalculate the apportionment.

state	population	population/*d*	seats
New York	7,264,183	37.606	38
Maine	694,466	3.595	3
Oklahoma	-	-	-
Total	74,562,608	386.004	386

The New State Paradox

In 1907, Oklahoma joined the union.

At around 1 million people, Oklahoma deserved five seats in the House.

Congress then added five seats, and used Hamilton's method to recalculate the apportionment.

All extra seats went to Oklahoma.

But New York lost a seat to Maine!

state	population	population/*d*	seats
New York	7,264,183	37.606	38
Maine	694,466	3.595	3
Oklahoma	-	-	-
Total	74,562,608	386.004	386

state	population	population/*d*	seats
New York	7,264,183	37.606	37
Maine	694,466	3.595	4
Oklahoma	1,000,000	5.175	5
Total	75,562,608	391.181	391

In response to these paradoxes Congress switched back to Webster's method.

In response to these paradoxes Congress switched back to Webster's method.

Webster's method is more impartial, but Hamilton's method was preferred by the large states.

Enter Willcox.

Walter Francis Willcox

1861 - 1964

Statistician at Cornell University.

Served as one of five chief statisticians for the US Census of 1900.

WALTER F. WILLCOX

After studying all the various apportionment methods,
I am convinced Webster's method is best.

Congress started leaning towards the Webster-Willcox method.

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But Ohio and Mississippi, which would have gotten an extra seat under Hamilton’s method, protested.

To keep everyone happy, in 1921 Congress kept Webster's method and increased the size of the House to 435.

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This number is still in place today.

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This number is still in place today.

But new ideas were needed.

Enter Hill.

Joseph Adna Hill

1860 - 1938

Statistician.

One of the authors of the *Method of Equal Proportions*, used to apportion representatives to states.

JOSEPH A. HILL

We should look at the number of people needed to get one representative.

What we called the *representation ratio*.

It doesn't seem fair to give state a representative per 50,000 people, and another state gets one per 70,000 people.

We should seek to minimize the relative difference between these quantities.

Minimizing Relative Differences

There are 20 seats for a population of 4 million, amounting, ideally, to $d = 200{,}000$ per seat.

The 20 seats are to be distributed among states 1 and 2, with populations 3,300,000 and 700,000, respectively.

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An allocation of 16 and 4 seats leads to a relative difference (i.e., ratio) of 1.18.

$d = 200{,}000$

state	population	population/d	seats	repr. ratio
1	3,300,000	16.5	16	206,250.00
2	700,000	3.5	4	175,000.00
Total	4,000,000	20	20	200,000.00

ratio of 1.18

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An allocation of 17 and 3 seats leads to a relative difference of 1.20.

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1	3,300,000	16.5	16	206,250.00
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state	population	population/d	seats	repr. ratio
1	3,300,000	16.5	17	194,117.65
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An allocation of 16 and 4 seats leads to a relative difference (i.e., ratio) of 1.18.

An allocation of 17 and 3 seats leads to a relative difference of 1.20.

The first allocation is more equal (1.18 < 1.20), and therefore preferred.

$d = 200{,}000$

state	population	population/d	seats	repr. ratio
1	3,300,000	16.5	16	206,250.00
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In general, we look for an apportionment where there's no possible reallocation from one state to another that reduces disparity.

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This involves reasoning over all pairs of states, and multiple divisors.

This requires a lot of computation.

Enter Huntington.

Edward Vermilye Huntington

1874 - 1952

Mathematician.

Big fan of Hill's *Method of Equal Proportions*, which would go on to be known as the *Huntington-Hill method*.

EDWARD V. HUNTINGTON

There’s a simpler way of thinking about Hill’s procedure.

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Consider first the following rounding function:

$$f(x) = \begin{cases} \lfloor x \rfloor, \text{ if } x < \sqrt{\lfloor x \rfloor \cdot \lceil x \rceil}, \\ \lceil x \rceil, \text{ if } x \geq \sqrt{\lfloor x \rfloor \cdot \lceil x \rceil}. \end{cases}$$

That is, we are rounding at the geometrical mean.

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That is, we are rounding at the geometrical mean.

Now fix a number *k* of seats.

Find a divisor *d* such that:

$$f\left(\frac{p_1}{d}\right) + \cdots + f\left(\frac{p_n}{d}\right) = k.$$

State *i* gets $f\left({}^{p_i}/_{d}\right)$ seats.

More generally, we can think of f as a rounding function that satisfies:

(*i*) $f(x) = x$, if x is an integer,

(*ii*) if $x \geq$ y, then $f(x) \geq f(y)$.

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(*i*) $f(x) = x$, if x is an integer,

(*ii*) if $x \geq$ y, then $f(x) \geq f(y)$.

We get a different apportionment method for every different rounding function.

Giving us the family of *divisor methods*.

THEOREM (HUNTINGTON, 1928)

A divisor method is the Huntington-Hill method if and only if for all states $i, j \in N$ such that ${p_i}/{k_i} \geq {p_j}/{k_j}$, it holds that:

$$\frac{p_i/k_i}{p_j/k_j} < \frac{p_j/(k_j-1)}{p_i/(k_i+1)}.$$

representation ratio

Huntington, E. V. (1928). The Apportionment of Representatives in Congress. *Transactions of the American Mathematical Society*, 30(1), 85–110.

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$d = 200{,}000$

state	population	population/d	seats	repr. ratio
i	p_i		k_i	p_i/k_i
j	p_j		k_j	p_j/k_j
Total	4,000,000	20	20	200,000.00

ratio of 1.18

$d = 200{,}000$

state	population	population/d	seats	repr. ratio
i	p_i		$k_i + 1$	$p_i/(k_i+1)$
j	p_j		$k_j - 1$	$p_j/(k_j-1)$
Total	4,000,000	20	20	200,000.00

ratio of 1.20

Huntington, E. V. (1928). The Apportionment of Representatives in Congress. *Transactions of the American Mathematical Society*, 30(1), 85–110.

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The larger states of New York, North Carolina and Virginia, who stood to lose one state, objected.

Deadlock resulted.

Deadlock resulted.

In 1921 Congress decided not to re-apportion the seats.

Deadlock resulted.

In 1921 Congress decided not to re-apportion the seats.

In direct violation to the Constitution (!).

WALTER F. WILLCOX

Mathematicians and statisticians are in favor of my method.

WALTER F. WILLCOX

Mathematicians and statisticians are in favor of my method.

EDWARD V. HUNTINGTON

Willcox's false description, supported by impressive charts and diagrams, is misleading.

Our method of equal proportions, with its simplicity, directness and intelligibility, leaves nothing to be desired.

After much acrimonious debate, both in Congress and scientific journals, the Huntington-Hill method prevailed.

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And stays on as the method used.

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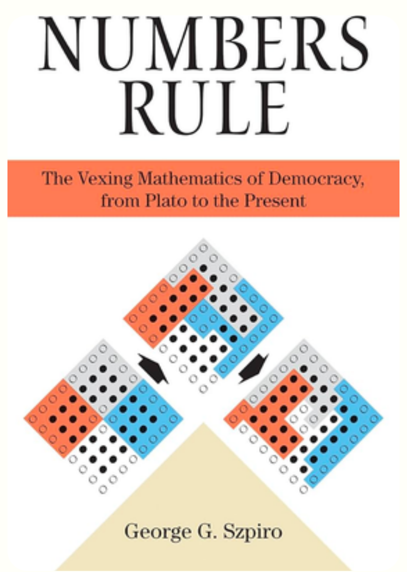
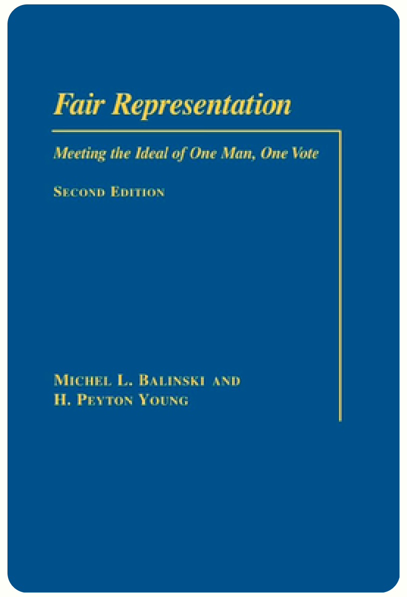
For now…

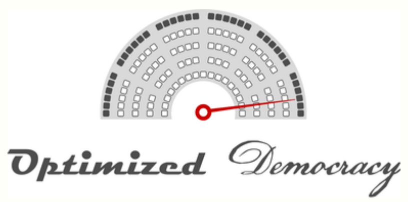
Read more here.



George Szpiro
Numbers Rule: The Vexing Mathematics of Democracy, from Plato to the Present
Princeton University Press
2010



Michel L. Balinski, H. Peyton Young
Fair Representation: Meeting the Ideal of One Man, One Vote
Brookings Institution Press
1982



Ariel Procaccia
Optimized Democracy
Harvard course
2021

Postscript.

Many of these apportionment methods were reinvented in Europe, and are used to this day to determine the constituency of Parliaments.

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In 1983, Balinski and Young showed that any reasonable apportionment rule is vulnerable to paradoxes.