LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

TWEAKING DEMOCRACY: INNOVATIONS IN DEMOCRATIC DECISION MAKING

Voting theory

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Voting is about aggregating information across a group.

When the goal is to take a decision that concerns the entire group.

And often involves finding a compromise between conflicting preferences.

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When the goal is to take a decision that concerns the entire group.

And often involves finding a compromise between conflicting preferences.

The stakes can be very high! Like determining who made the better Queen…

CLAIRE FOY

People seem to be very fond of ranking things, though their opinions and wants can differ significantly.

OLIVIA COLMAN

Nonetheless, for giving out awards, or recommendations, we may want to combine their rankings into a single outcome.

IMELDA STAUNTON

The outcome could range in terms of structure, from another ranking, to a single winner, or a set of winners.

472	380	96	19	29	9
Foy	*Foy*	*Colman*	*Colman*	*Staunton*	*Staunton*
Colman	*Staunton*	*Foy*	*Staunton*	*Foy*	*Colman*
Staunton	*Colman*	*Staunton*	*Foy*	*Colman*	*Foy*

Foy, Colman, Staunton?
{Foy, Colman}?
Foy?

What is your order for the actresses who played Queen Elizabeth the best?. Reddit.

KENNETH ARROW

Voting theory underwrites the science of political elections.

And is thus important in a democratic society.

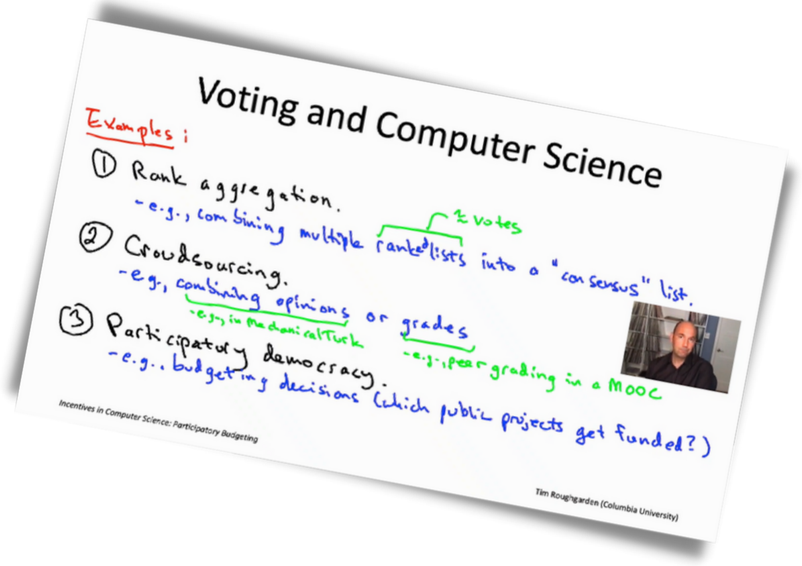
TIM ROUGHGARDEN

But is also important in many applications of AI and Computer Science.

Rank Aggregation.

Crowdsourcing.

Participatory budgeting.



Durand, F., Macé, A., Núñez, M. (2021). A representative voting system: Approval Voting. Paris School of Economics.
Roughgardern, T. (2021). Incentives in Computer Science. YouTube.

A Potpourri of Voting Rules

agents, or voters	$N = \{1, \ldots, n\}$
alternatives, or candidates	$A = \{a, b, c, \ldots\}, \lvert A\rvert = m$
preference order of voter i	$\succ_i$, linear order on alternatives
set of all possible preferences	$L = \{\succ \mid \succ \text{ is a linear order on } A\}$
preference profile	$(\succ_1, \ldots, \succ_n) \in L^n$
social choice function	$F \colon L^n \to 2^A \setminus \{\emptyset\}$
resolute social choice function	$F \colon L^n \to A$
social welfare function	$F \colon L^n \to L$

Many voting rules can sometimes result in ties.

If we need to avoid this we can use some tie-breaking rule, or, sometimes, assume that the number of voters is odd.

DEFINITION (MAJORITY)

Every agent reports their top choice. The winners are the alternatives that get at least half of the votes.

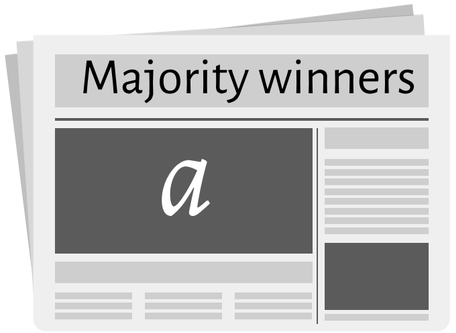
55	45
a	b
b	a

LATEST NEWS

Used

… whenever a group needs to choose between a and b.

55	45
a	b
b	a



Pros

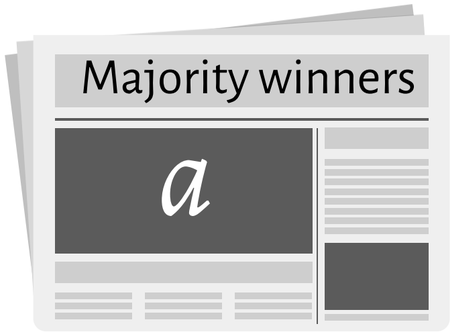
… simple, intuitive.

Cons

Used

... whenever a group needs to choose between a and b.

55	45
a	b
b	a



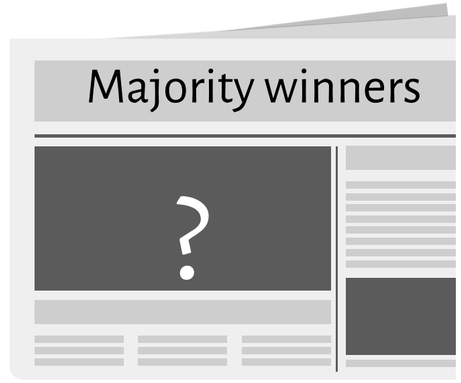
Pros

... simple, intuitive.

Cons

... doesn't always deliver an answer for more than two alternatives.

34	33	33
a	b	c
b	c	a
c	a	b

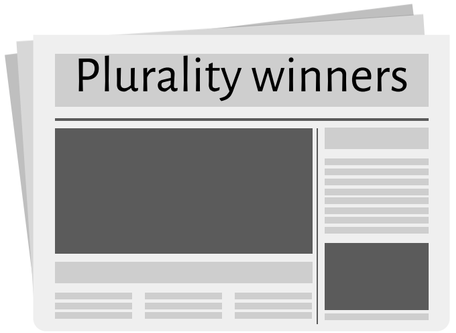


DEFINITION (PLURALITY*)

Every agent reports their top choice. The winners are the alternatives that get the most votes.

*also known as *first-past-the-post* (*FPTP*)

34	33	33
a	*b*	*c*
b	*c*	*b*
c	*a*	*a*



Used

… to elect representatives (e.g., members of parliament, mayors) in the UK, the US, Canada, India.

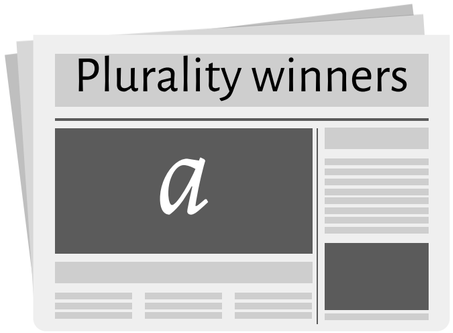
… all in all, for local and national elections in 43 of the 193 countries of the United Nations.

Pros

… simple, works for any number of alternatives.

Cons

34	33	33
a	*b*	*c*
b	*c*	*b*
c	*a*	*a*



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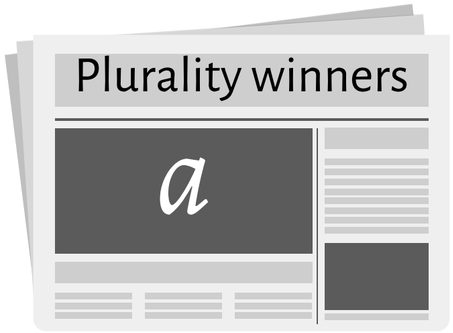
Pros

… simple, works for any number of alternatives.

Cons

… may produce very bad results.

34	33	33
a	*b*	*c*
b	*c*	*b*
c	***a***	***a***



a is hated by two thirds of the electorate!

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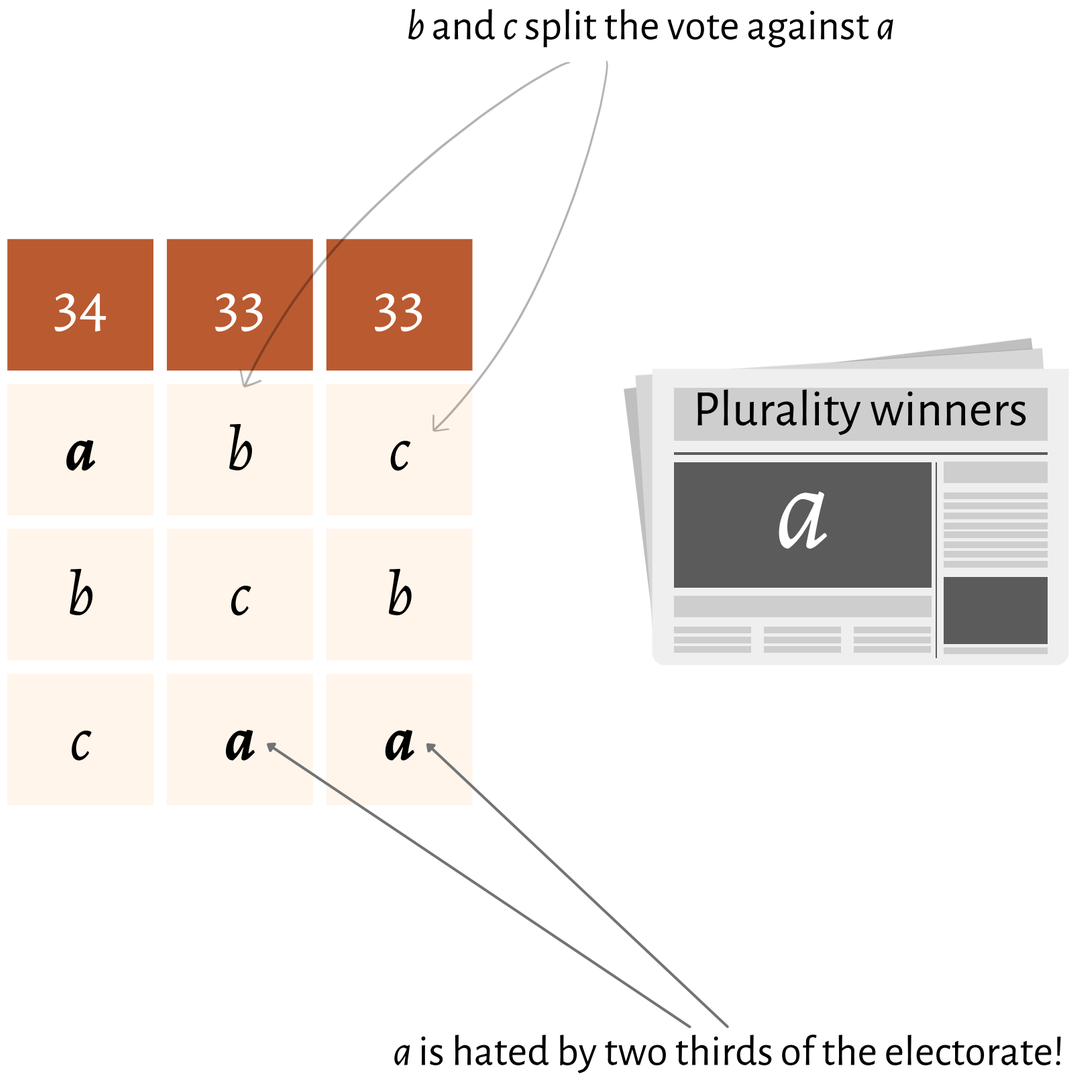
Pros

… simple, works for any number of alternatives.

Cons

… may produce very bad results.



Used

… to elect representatives (e.g., members of parliament, mayors) in the UK, the US, Canada, India.

… all in all, for local and national elections in 43 of the 193 countries of the United Nations.

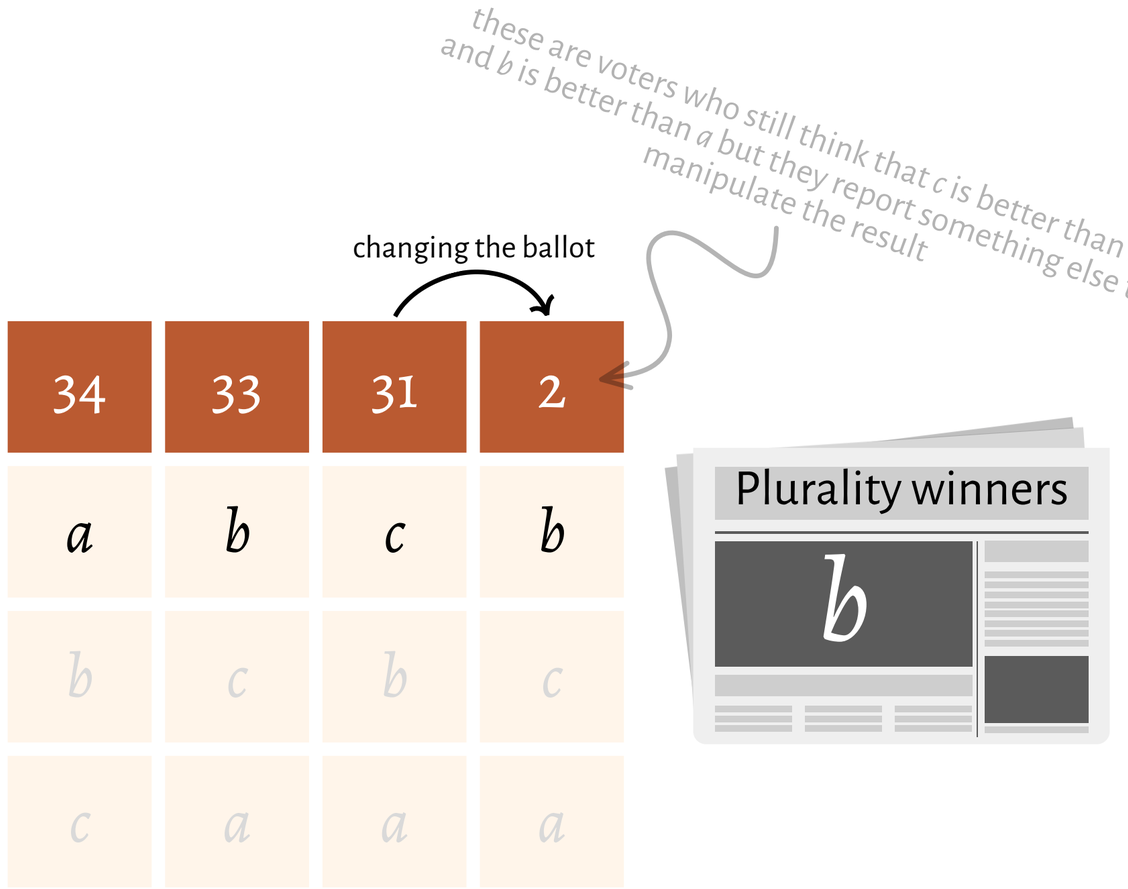
Pros

… simple, works for any number of alternatives.

Cons

… may produce very bad results.

… and encourages tactical voting.



a is hated by two thirds of the electorate!

A similar thing happens if the population is divided into districts.

Suppose *a*, *b* and *c* are parties and each district gets a seat on the city council.

WEST

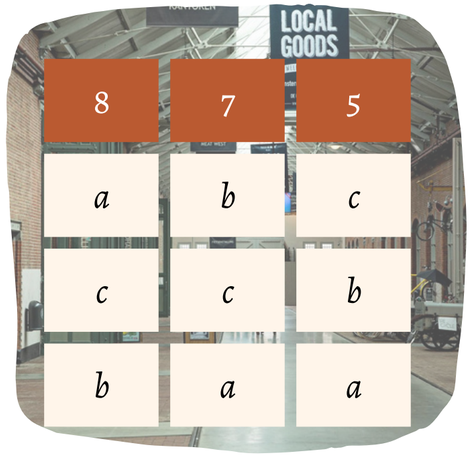
ZUID

OOST

CENTRUM

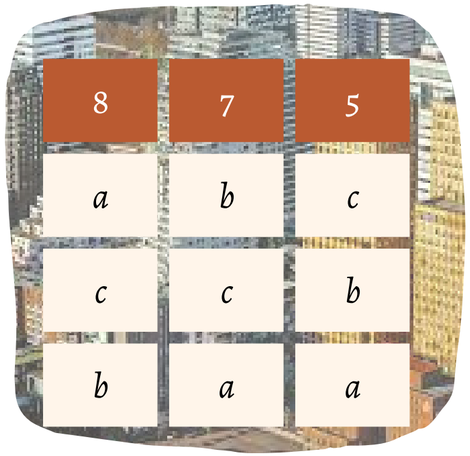
NOORD

WEST



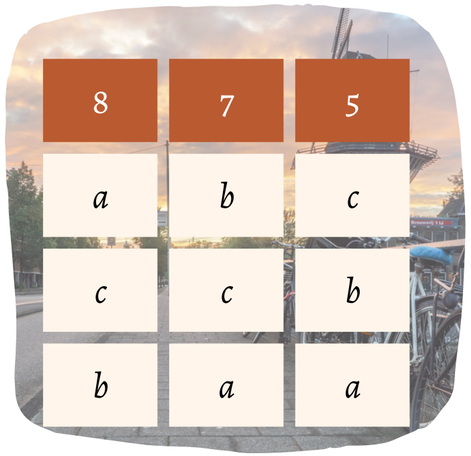
8	7	5
a	b	c
c	c	b
b	a	a

ZUID



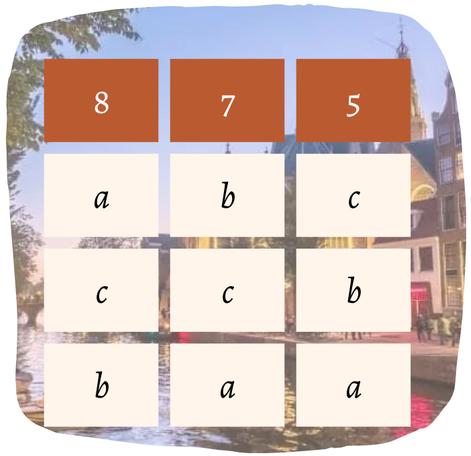
8	7	5
a	b	c
c	c	b
b	a	a

OOST



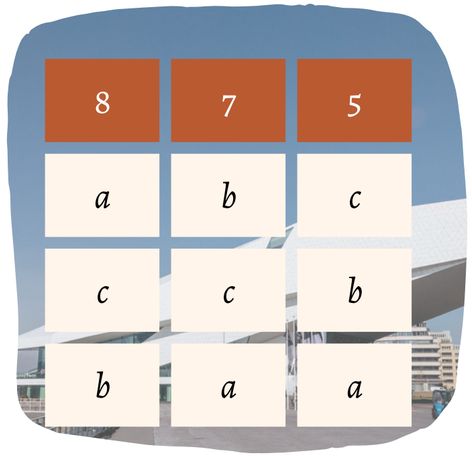
8	7	5
a	b	c
c	c	b
b	a	a

CENTRUM



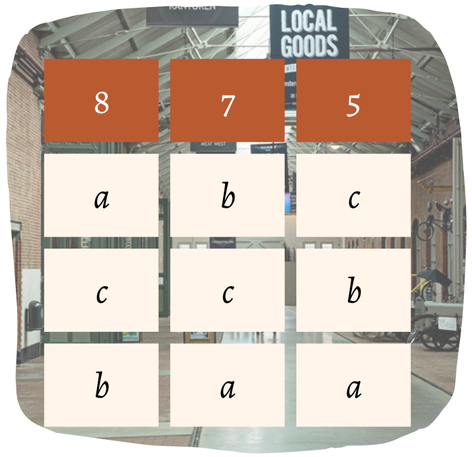
8	7	5
a	b	c
c	c	b
b	a	a

NOORD



8	7	5
a	b	c
c	c	b
b	a	a

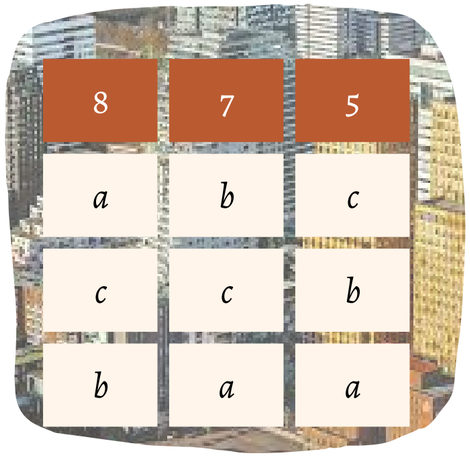
WEST



8	7	5
a	*b*	*c*
c	*c*	*b*
b	*a*	*a*

a

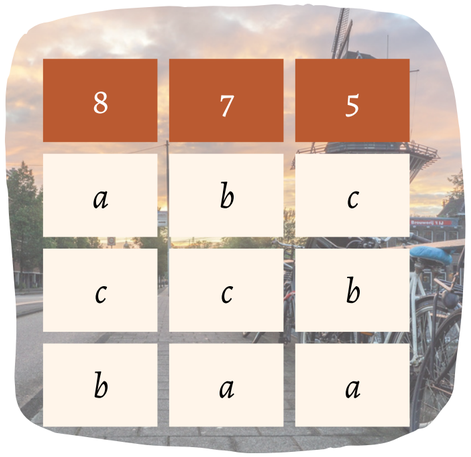
ZUID



8	7	5
a	*b*	*c*
c	*c*	*b*
b	*a*	*a*

a

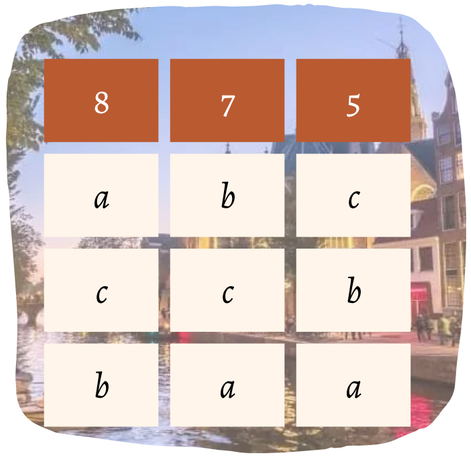
OOST



8	7	5
a	*b*	*c*
c	*c*	*b*
b	*a*	*a*

a

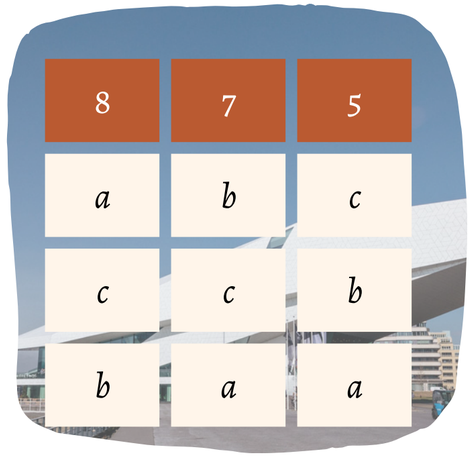
CENTRUM



8	7	5
a	*b*	*c*
c	*c*	*b*
b	*a*	*a*

a

NOORD



8	7	5
a	*b*	*c*
c	*c*	*b*
b	*a*	*a*

a

THIS JUST IN

With plurality party *a* gets all five seats, despite being ranked last by 60% of the voters!

MAURICE DUVERGER

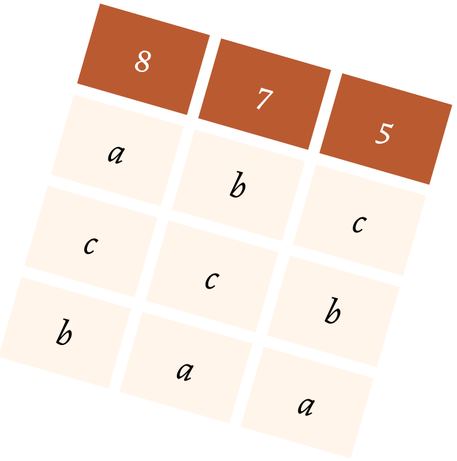
In the long run, *c* loses support as more and more voters migrate towards *b*.

In general, smaller parties get squeezed out of power.

Because no one bothers to vote for them.

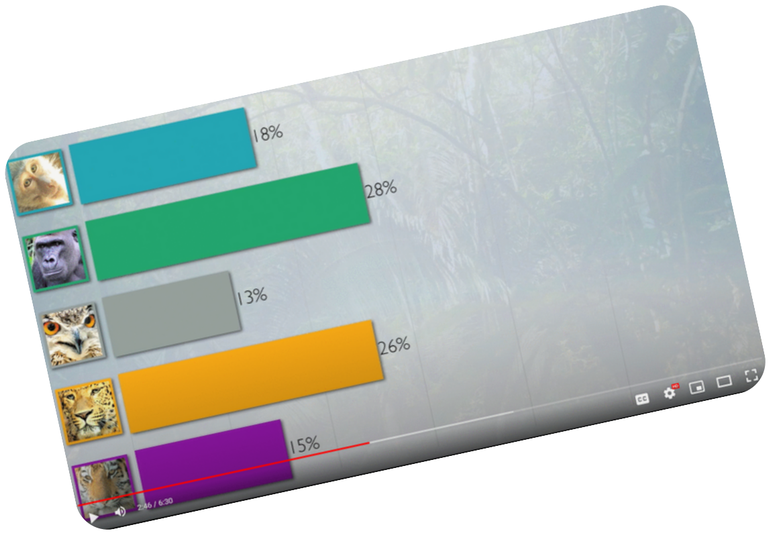
Because they don't win anyway.

A single-ballot plurality-rule election structured within single-member districts tends to favor a two party system.



CGP GREY

This is known nowadays as *Duverger's Law*.

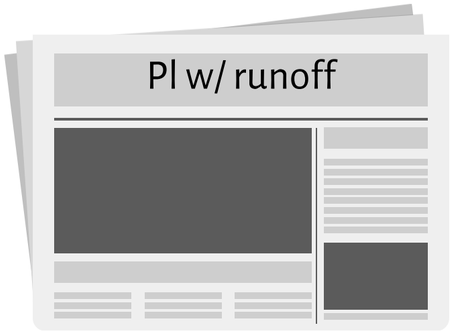


Duverger, M. (1964). Political parties: their organization and activity in the modern state. London: Methuen.
CGP Grey (2011). Minority Rule: First Past the Post Voting. YouTube.

DEFINITION (PLURALITY WITH RUNOFF)

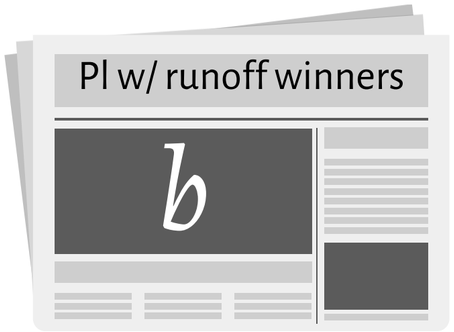
Every agent reports their top choice. If there is a candidate that gets a majority of the votes, they are declared the winner. If not, hold an extra round of voting between the two candidates that get the most votes. The majority winner at this round is declared the winner.

40	35	25
a	*b*	*c*
c	*c*	*b*
b	*a*	*a*



40	35	25
a	*b*	*c*
c	*c*	*b*
b	*a*	*a*

40	35	25
a	*b*	*c*
c	*c*	*b*
b	*a*	*a*



Used

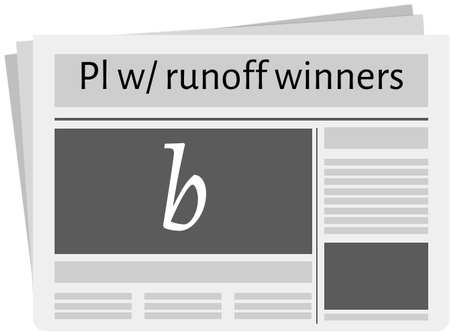
... to elect presidents in France, Romania, etc.

Pros

... reduces need for tactical voting.

40	35	25
a	*b*	*c*
c	*c*	*b*
b	*a*	*a*

40	35	25
a	*b*	*c*
c	*c*	*b*
b	*a*	*a*



Used

… to elect presidents in France, Romania, etc.

Pros

… reduces need for tactical voting.

Cons

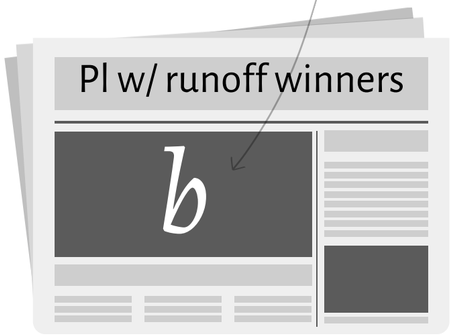
… though does not eliminate it: sometimes have to lie to make sure a preferred alternative makes it to the second round.

… and may still produce very bad results.

40	35	25
a	*b*	*c*
c	*c*	*b*
b	*a*	*a*

65% prefer *c* to *b*

40	35	25
a	*b*	*c*
c	*c*	*b*
b	*a*	*a*



Turns out plurality, a super-popular voting rule, is not very good.

It allows for alternatives to get elected even when there is some other alternative that a majority thinks is better.

And having a runoff does not fix the problem.

CONDORCET

I have an idea!

Winners should be the alternatives that beat every other alternative in a head-to-head contest.

MARQUES, AKA MKBHD

Kind of how we do our end-of-year mobile phone rankings.

See the 2022 smartphone awards video!

DEFINITION (THE CONDORCET RULE)

We write $n(x, y)$ for the number of agents who prefer alternative x to alternative y.

A *Condorcet winner* is an alternative x^* such that $n(x^*, y) > n(x^*, y)$, for any (other) alternative y.

4	3	3	3
a	*b*	*c*	*d*
b	*c*	*d*	*c*
c	*d*	*b*	*b*
d	*a*	*a*	*a*

4	3	3	3
a	b	c	d
b	c	d	c
c	d	b	b
d	a	a	a

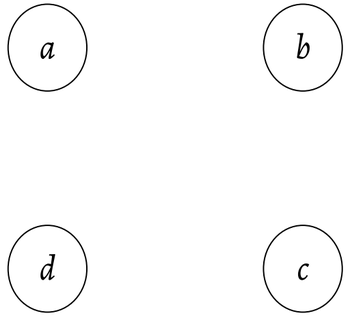
$n(a, b) = 4$

$n(b, a) = 9$

4	3	3	3
a	b	c	d
b	c	d	c
c	d	b	b
d	a	a	a

$n(a, b) = 4$

$n(b, a) = 9$

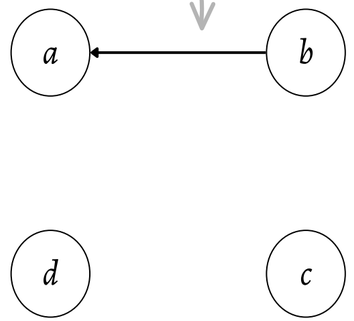


4	3	3	3
a	b	c	d
b	c	d	c
c	d	b	b
d	a	a	a

more people prefer b to a than the other way around

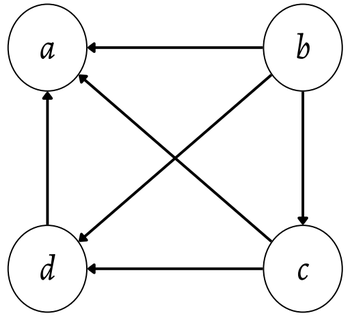
$n(a, b) = 4$

$n(b, a) = 9$



4	3	3	3
a	b	c	d
b	c	d	c
c	d	b	b
d	a	a	a

$n(a, b) = 4$
$n(b, a) = 9$
$n(b, c) = 7$
$n(c, b) = 6$
$n(b, d) = 7$
$n(d, b) = 6$



4	3	3	3
a	b	c	d
b	c	d	c
c	d	b	b
d	a	a	a

$n(a, b) = 4$
$n(b, a) = 9$
$n(b, c) = 7$
$n(c, b) = 6$
$n(b, d) = 7$
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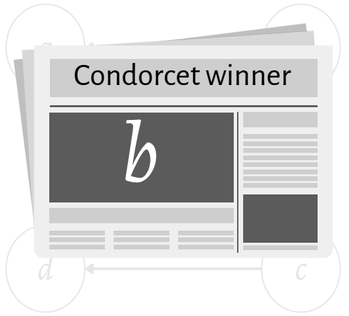
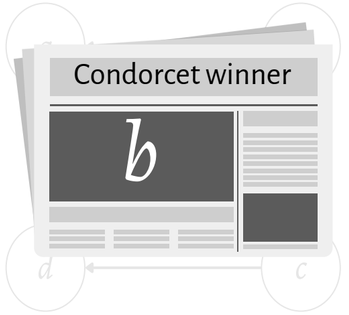


Used

... in this form, nowhere (will see in a moment why).

4	3	3	3
a	b	c	d
b	c	d	c
c	d	b	b
d	a	a	a

$n(a, b) = 4$
$n(b, a) = 9$
$n(b, c) = 7$
$n(c, b) = 6$
$n(b, d) = 7$
$n(d, b) = 6$



Pros

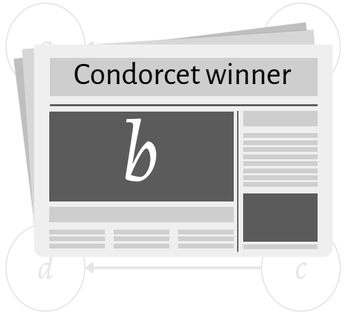
... makes sense.

Used

... in this form, nowhere (will see in a moment why).

4	3	3	3
a	b	c	d
b	c	d	c
c	d	b	b
d	a	a	a

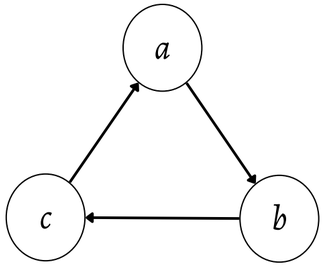
$n(a, b) = 4$
$n(b, a) = 9$
$n(b, c) = 7$
$n(c, b) = 6$
$n(b, d) = 7$
$n(d, b) = 6$



Pros

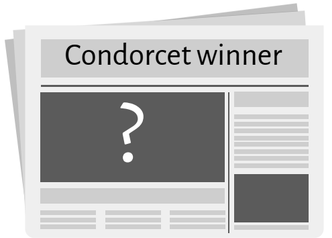
... makes sense.

1	1	1
a	b	c
b	c	a
c	a	b



Cons

... doesn't always exist (!).



DEFINITION (CONDORCET CONSISTENT RULES)

A voting rule is *Condorcet consistent* if it selects the Condorcet winner, when it exists.

CONDORCET

The Condorcet method needs to be tweaked to make sure it returns an answer when a Condorcet winner does not exist.

As such, Condorcet consistent rules are used in practice.

The Wikimedia Foundation used the Schulze method to elect its Board of Trustees until 2013.

The Pirate Party of Sweden uses the Schulze method for its primaries.

The Debian project uses the Schulze method for internal referendums and to elect its leader.

- Black
- Copeland
- Dodgson
- Kemeny
- Minimax
- Nanson
- Ranked pairs
- Schulze

MARQUES, AKA MKBHD

Note that for smartphones, ain’t nobody got time to rank all possible pairs of phones.

MARQUES, AKA MKBHD

Note that for smartphones, ain’t nobody got time to rank all possible pairs of phones.

So what we implement is a restricted set of comparisons, where winners from one pair go on to be pitted against winners from a different pair, and so on.

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Note that for smartphones, ain't nobody got time to rank all possible pairs of phones.

So what we implement is a restricted set of comparisons, where winners from one pair go on to be pitted against winners from a different pair, and so on.

Also called a *tournament*.

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Note that for smartphones, ain't nobody got time to rank all possible pairs of phones.

So what we implement is a restricted set of comparisons, where winners from one pair go on to be pitted against winners from a different pair, and so on.

Also called a *tournament*.

Yes, like in sports.

Looking at head-to-head contests does not always result in a meaningful ranking.

As there can be majority cycles.

One suggestion is that we're not using all the information available in the profile.

BORDA

I have an idea!

DEFINITION (THE BORDA RULE)

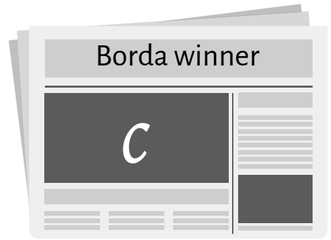
Every voter i gives to alternative x a score of $m - pos_i(x)$, called *the Borda score*, where $pos_i(x) \in \{1, \ldots, m\}$ is the position of x in i's preference order $\succ_i$.

The *Borda winners* are the alternatives with the highest overall score, i.e., that maximize the sum of the Borda scores over all voters.

4	3	3	3
a	b	c	d
b	c	d	c
c	d	b	b
d	a	a	a

Borda scores

a: 12
b: 23
c: 25
d: 18



Used

… in the National Assembly of Slovenia,
Icelandic parliamentary elections.

Pros

… Borda winners always exist.

4	3	3	3
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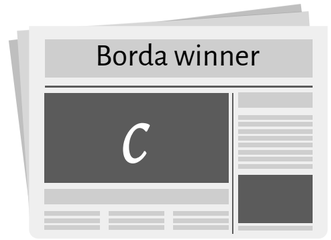

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Pros

… Borda winners always exist.

Cons

… sensitive to the introduction/removal of irrelevant alternatives.

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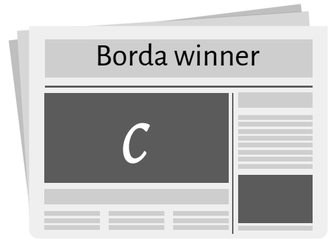
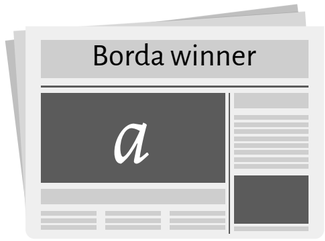


35	33	32
a	b	c
c	a	b
b	c	a



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a	*b*	*c*	*d*
b	*c*	*d*	*c*
c	*d*	*b*	*b*
d	*a*	*a*	*a*

Borda scores

a: 12
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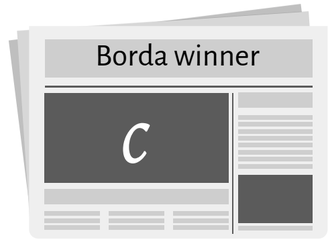
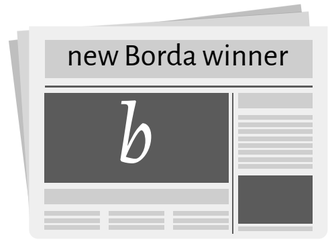


35	33	32
a	*b*	*c*
c	*a*	*b*
b	*c*	*a*

Candidate *c*, who has no chance of winning, acts as a spoiler for *b*



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Pros

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Cons

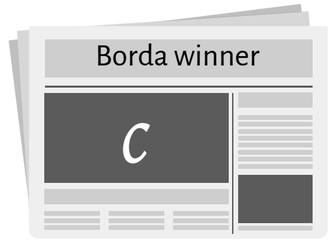
… sensitive to the introduction/removal of irrelevant alternatives.

… can be manipulated by strategic agents.

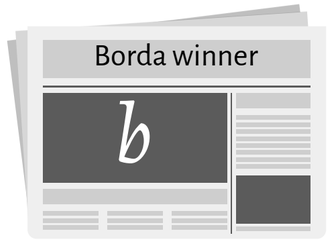
4	3	3	3
a	b	c	d
b	c	d	c
c	d	b	b
d	a	a	a

Borda scores

a: 12
b: 23
c: 25
d: 18



1	1	1
b	b	a
a	a	b
c	c	c
d	d	d



Used

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Pros

... Borda winners always exist.

Cons

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... can be manipulated by strategic agents.

4	3	3	3
a	*b*	*c*	*d*
b	*c*	*d*	*c*
c	*d*	*b*	*b*
d	*a*	*a*	*a*

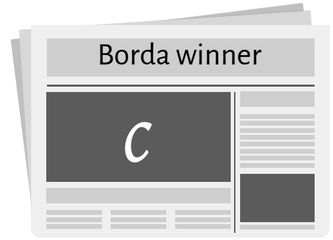
changing the ballot

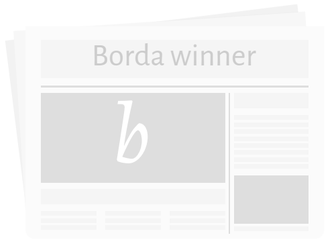
1	1	1	1
b	*b*	*a*	*a*
a	*a*	*b*	*c*
c	*c*	*c*	*d*
d	*d*	*d*	*b*

By pushing b down in their ranking, the voter makes a win. To their advantage.

Borda scores

a: 12
b: 23
c: 25
d: 18





BORDA

My rule is intended for honest people!

The Borda rule is one instance of a broader class of rules: *scoring rules*.

DEFINITION (SCORING RULE)

A *scoring rule* uses a scoring vector $\boldsymbol{s} = (s_1, \ldots, s_n)$, with $s_1 \geq \ldots \geq s_n$ and $s_1 > s_n$, to assign score s_j to candidate in position j of voter i's ranking.

For every alternative, we add up the score across all voters.

The winners are the alternatives with the highest overall score.

BORDA

For the Borda rule the scores are:

$$\boldsymbol{s} = (m-1, m-2, \ldots, 0)$$

ME

For plurality:

$$\boldsymbol{s} = (1, 0, \ldots, 0)$$

RUSS KUN, PRESIDENT OF NAURU

In Nauru we use the Dowdall scoring vector:

$$\boldsymbol{s} = \left(1, \frac{1}{2}, \ldots, \frac{1}{m}\right)$$

ME

And then, of course:

$$\boldsymbol{s} = (12, 10, 8, 7, 6, 5, 4, 3, 2, 1)$$

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$$\boldsymbol{s} = (m-1, m-2, \ldots, 0)$$

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For plurality:

$$\boldsymbol{s} = (1, 0, \ldots, 0)$$

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ME

And then, of course:

$$\boldsymbol{s} = (12, 10, 8, 7, 6, 5, 4, 3, 2, 1)$$

DEFINITION (SINGLE TRANSFERABLE VOTE, OR STV*)

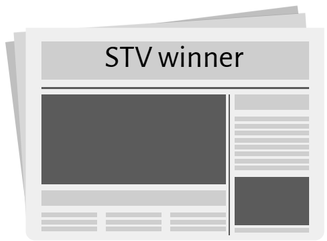
The rule proceeds in rounds. At every round, the alternative that shows up on top least often is eliminated (if several, use a tie-breaking rule).

The *STV winner* is the last alternative left standing.

*also known as *instant run-off voting*, *Alternative Vote* (*AV*), *ranked choice voting*

round 0

4	3	3	3
a	*b*	*c*	*d*
b	*c*	*d*	*c*
c	*d*	*b*	*b*
d	*a*	*a*	*a*

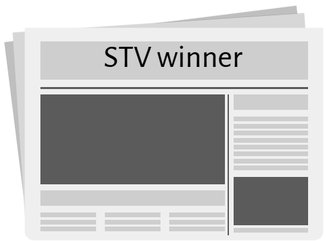


round 0

4	3	3	3
a	*b*	*c*	*d*
b	*c*	*d*	*c*
c	*d*	*b*	*b*
d	*a*	*a*	*a*

round 1

4	3	3	3
a	*b*	*c*	*d*
b	*c*	*d*	*c*
c	*d*	*b*	*b*
d	*a*	*a*	*a*



round 0

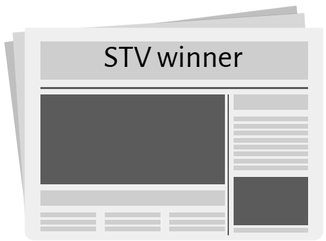
4	3	3	3
a	*b*	*c*	*d*
b	*c*	*d*	*c*
c	*d*	*b*	*b*
d	*a*	*a*	*a*

round 1

4	3	3	3
a	*b*	*c*	*d*
b	*c*	*d*	*c*
c	*d*	*b*	*b*
d	*a*	*a*	*a*

round 2

4	3	3	3
a	*b*	*c*	*d*
b	*c*	*d*	*c*
c	*d*	*b*	*b*
d	*a*	*a*	*a*



round 0

4	3	3	3
a	*b*	*c*	*d*
b	*c*	*d*	*c*
c	*d*	*b*	*b*
d	*a*	*a*	*a*

round 1

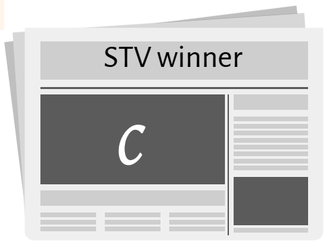
4	3	3	3
a	*b*	*c*	*d*
b	*c*	*d*	*c*
c	*d*	*b*	*b*
d	*a*	*a*	*a*

round 2

4	3	3	3
a	*b*	*c*	*d*
b	*c*	*d*	*c*
c	*d*	*b*	*b*
d	*a*	*a*	*a*

round 3

4	3	3	3
a	*b*	*c*	*d*
b	*c*	*d*	*c*
c	*d*	*b*	*b*
d	*a*	*a*	*a*



Used

... to elect members of the Australian House of Representatives, the President of India, the President of Ireland, governors in Alaska.

... to choose the Academy Award for Best Picture.

Pros

... less vulnerable to tactical voting.

Cons

... not Condorcet consistent.

... somewhat complicated to get across to people.

round 0

4	3	3	3
a	*b*	*c*	*d*
b	*c*	*d*	*c*
c	*d*	*b*	*b*
d	*a*	*a*	*a*

round 1

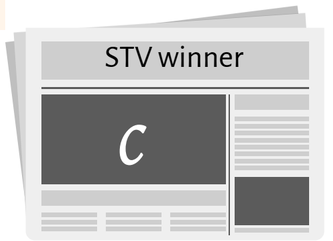
4	3	3	3
a	*b*	*c*	*d*
b	*c*	*d*	*c*
c	*d*	*b*	*b*
d	*a*	*a*	*a*

round 2

4	3	3	3
a	*b*	*c*	*d*
b	*c*	*d*	*c*
c	*d*	*b*	*b*
d	*a*	*a*	*a*

round 3

4	3	3	3
a	*b*	*c*	*d*
b	*c*	*d*	*c*
c	*d*	*b*	*b*
d	*a*	*a*	*a*



TIMOTHY GOWERS

Under STV, a vote for a minor party is not a wasted vote.

DAVID CAMERON

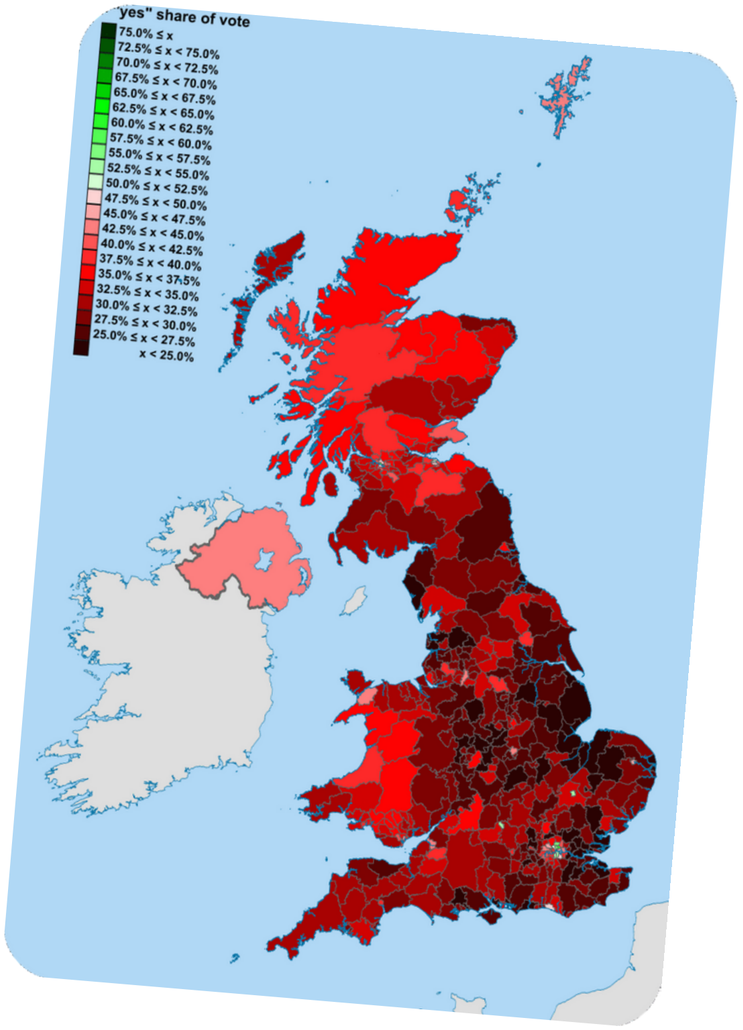
STV is undemocratic, obscure, unfair and crazy.

SARAH PALIN

A new crazy, convoluted, confusing system.

In the August 2022 US House election it failed to elect the Condorcet winner!

Which, admittedly, wasn't me.



Gowers, T. (2011). Is AV better than FPTP?. Gower's Weblog.

2011 United Kingdom Alternative Vote referendum. Wikipedia.

Clark, T. (2011). 10 reasons the AV referendum was lost. The Guardian.

David Cameron: Alternative vote is a 'crazy, undemocratic system'. BBC News.

Graham-Squire, A., McCune, D. (2022). A Mathematical Analysis of the 2022 Alaska Special Election for US House. arXiv.

DEFINITION (APPROVAL VOTING)

Ballots are subsets of alternatives: the ones voters approve of.

Approval winners are alternatives that have the most approvals.

	a	*b*	*c*	*d*	*e*
voter 1	✓		✓	✓	
voter 2				✓	
voter 3	✓	✓			
voter 4			✓	✓	✓
voter 5		✓			

Used

… in municipal elections in Fargo, North Dakota and St Louis, Missouri.

Pros

… is not vulnerable to tactical voting.

… strikes a good balance between expressiveness and difficulty, prevents minor party candidates from being spoilers.

	a	*b*	*c*	*d*	*e*
voter 1	✓		✓	✓	
voter 2				✓	
voter 3	✓	✓			
voter 4			✓	✓	✓
voter 5		✓			

BORDA

We've seen a bunch of voting rules.

Majority, various versions of plurality, Borda, approval.

BORDA

We've seen a bunch of voting rules.

Majority, various versions of plurality, Borda, approval.

CONDORCET

Rules based on the Condorcet criterion!

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There's also a lot more out there!

Like quadratic voting, score voting, threshold rules, selection by lot, ...

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CONDORCET

Rules based on the Condorcet criterion!

BORDA

There's also a lot more out there!

Like quadratic voting, score voting, threshold rules, selection by lot, …

CONDORCET

And we've seen some problems that voting rules run into.

Disrespecting majorities, encouraging voters to misreport their preferences, sensitivity to irrelevant alternatives…

Let's vote!

http://whale.imag.fr/polls/vote/fdce351e-ea96-408f-b1fa-1390806e9eb8

So, um… what voting rule *should* we use?

JEAN-FRANCOIS LASLIER

Experts have different opinions as to which is the best voting procedure.

But it seems like they agree that Plurality is the worst.

And, for the most part, think Approval is the best.

Admittedly, results were aggregated using Approval…

Table 13.2 Approval scores

Voting rule		Approvals	Approving percentage
Approval voting	App	15	68.18
Alternative vote	Alt	10	45.45
Copeland	Cop	9	40.91
Kemeny	Kem	8	36.36
Two-round majority	2R	6	27.27
Coombs	Coo	6	27.27
Simpson	Sim	5	22.73
Majority judgement	Bal	5	22.73
Borda	Bor	4	18.18
Black	Bla	3	13.64
Range voting	RV	2	9.09
Nanson	Nan	2	9.09
Leximin	Lex	1	4.54
Top-cycle	TC	1	4.54
Uncovered set	UC	1	4.54
Fishburn		0	0
Untrapped set		0	0
Plurality		0	0

Laslier, J.-F. (2012). And the Loser Is… Plurality Voting. In D. S. Felsenthal & M. Machover (Eds.), *Electoral Systems: Paradoxes, Assumptions, and Procedures* (pp. 327–351). Springer Berlin Heidelberg.

Characterizations and Impossibilities

So many voting rules: which one is the best?

We need some general principles to distinguish between voting rules.

KENNETH ARROW

Let's take the point of view of someone who wants to design a voting rule from scratch, and think about what properties, or *axioms*, we'd want the voting rule to satisfy.

agents, or voters	$N = \{1, \ldots, n\}$
alternatives, or candidates	$A = \{a, b, c, \ldots\}, \lvert A\rvert = m$
preference order of voter i	$\succ_i$, linear order on alternatives
set of all possible preferences	$L = \{\succ \mid \succ \text{ is a linear order on } A\}$
preference profile	$\boldsymbol{R} = (\succ_1, \ldots, \succ_n) \in L^n$
social choice function	$F \colon L^n \to 2^A \setminus \{\emptyset\}$
resolute social choice function	$F \colon L^n \to A$
social welfare function	$F \colon L^n \to L$

The first axiom we look at is *anonymity*.

It says that the order in which we arrange the voters does not matter for the final result.

AXIOM (ANONYMITY)

A voting rule F satisfies *Anonymity* if, for any permutation σ of the set N of voters, it holds that:

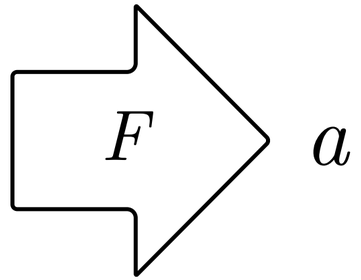
$$F(\succ_1, \ldots, \succ_n) = F\left(\succ_{\sigma(1)}, \ldots, \succ_{\sigma(n)}\right)$$

Anonymity requires invariance under permutations of the voters in the profile.

Permutation here is σ(Ann) = Bob, σ(Bob) = Cat, σ(Cat) = Ann

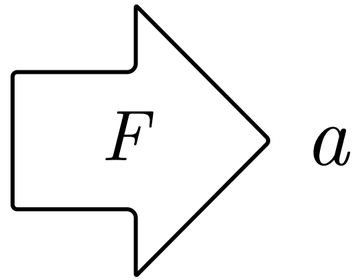
If

Ann	Bob	Cat
a	a	b
b	c	a
c	b	c



then

Bob	Cat	Ann
a	b	a
c	a	b
b	c	c



The next property is *neutrality*.

It says that the names we give to alternatives do not matter. A rose by any other name…

AXIOM (NEUTRALITY)

A voting rule F satisfies *Neutrality* if, for any permutation σ of the set A of alternatives, it holds that:

$$\sigma\big(F(\succ_1, \ldots, \succ_n)\big) = F\left(\sigma\left(\succ_1\right), \ldots, \sigma\left(\succ_n\right)\right)$$

Every alternative is replaced with its image under σ.

Neutrality requires that permutations of the alternatives in the profile are reflected by permutations of the alternatives in the result.

$\sigma(a) = b,$
$\sigma(b) = c,$
$\sigma(c) = a$

If

Ann	Bob	Cat
a	a	b
b	c	a
c	b	c

F → a

then

Ann	Bob	Cat
b	b	c
c	a	b
a	c	a

F → b

And now for something a bit more involved: *positive responsiveness*.

It says, roughly, that increased support for some alternative has the power to break a tie in favor of that alternative.

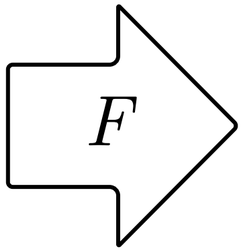
AXIOM (POSITIVE RESPONSIVENESS)

A social choice function F satisfies *Positive Responsiveness* if, for any distinct profiles $\boldsymbol{R}$ and $\boldsymbol{R}'$ and alternative x^*, we have that $\boldsymbol{R}$ and $\boldsymbol{R}'$ are the same except that in $\boldsymbol{R}'$ some voters move x^* up some positions in their preference rankings, then it holds that if $x^* \in F(\boldsymbol{R})$, then $F(\boldsymbol{R}') = \{x^*\}$.

If in $\boldsymbol{R}'$ some voters raise x^*, while leaving everything else untouched, then x^* goes from being a (possibly tied) winner to the unique winner.

If

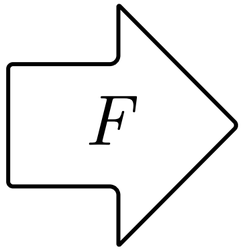
Ann	Bob	Cat	Dov
a	a	b	b
b	b	a	a



$\{a,\ b\}$

then

Ann	Bob	Cat	Dov
a	a	a	b
b	b	b	a



$\{a\}$

KENNETH O. MAY

For two alternatives, it turns out that these properties are satisfied only by the majority voting rule.

Note that when there are only two alternatives, the majority rule is well-defined.

THEOREM (MAY, 1952)

If there are only two alternatives, then the only social choice function that satisfies Anonymity, Neutrality and Positive Responsiveness is the majority rule.

May, K.O. (1952). A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision. *Econometrica*. 20(4):680–684.

KENNETH O. MAY

For two alternatives we can't do better than using majority.

And note that when there are only two alternatives, all the voting rules we've looked at so far are equivalent to the majority rule.

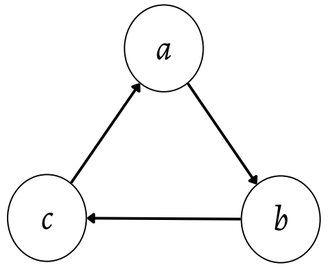
Now for more than two alternatives...

CONDORCET

We know that majority comparisons can get us into trouble with cycles.

But maybe there's some other clever way to combine preferences into a coherent social ranking.

1	1	1
a	*b*	*c*
b	*c*	*a*
c	*a*	*b*



KENNETH O. MAY

For two alternatives we can't do better than using majority.

And note that when there are only two alternatives, all the voting rules we've looked at so far are equivalent to the majority rule.

Now for more than two alternatives...

CONDORCET

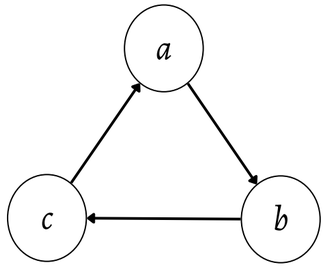
We know that majority comparisons can get us into trouble with cycles.

But maybe there's some other clever way to combine preferences into a coherent social ranking.

KENNETH ARROW

Ah about that...

1	1	1
a	*b*	*c*
b	*c*	*a*
c	*a*	*b*



KENNETH ARROW

For the next result we will focus on *social welfare functions*: voting rules that return a ranking of the alternatives.

Let's write down some more reasonable properties.

VILFREDO PARETO

If everyone thinks some alternative is better than another, then this should be reflected in the result.

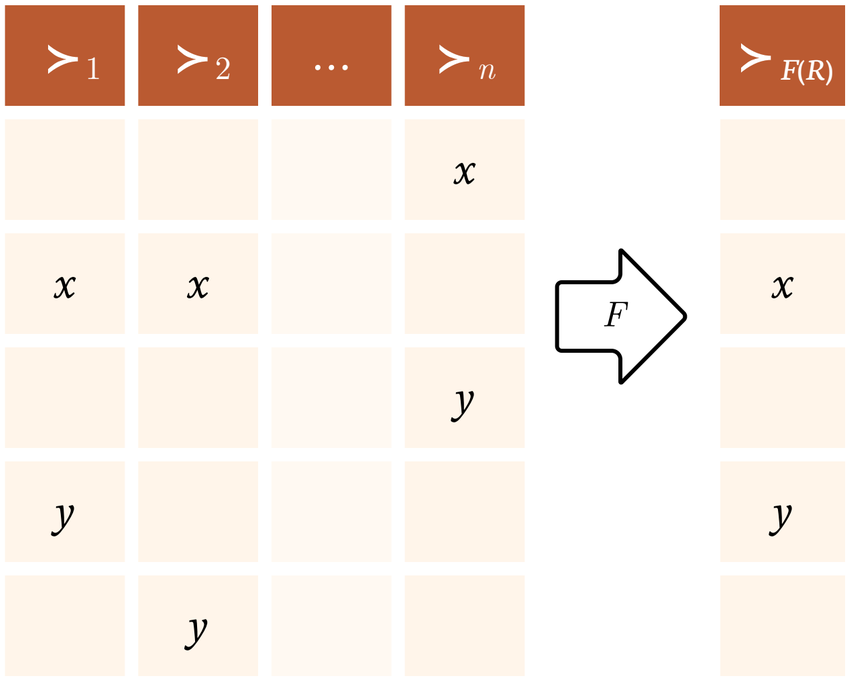
AXIOM (PARETO EFFICIENCY)

A social welfare function F satisfies *Pareto Efficiency* if, for any alternatives x and y, it holds that if $x \succ_i y$, for every voter $i \in N$, then $x \succ_{F(\boldsymbol{R})} y$.

society's ranking

There is unanimous agreement that *x* is better than *y*.

Hence *x* is ranked above *y* in the aggregated ranking.



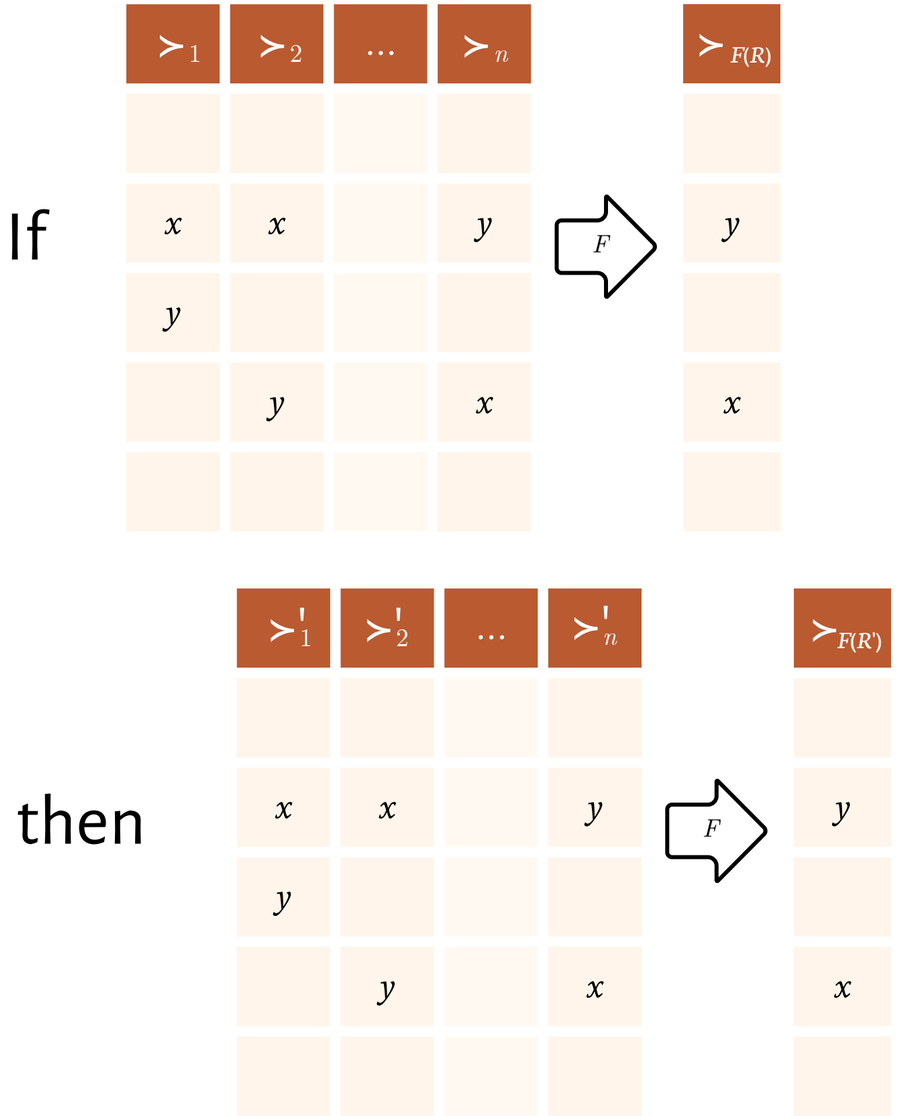
KENNETH ARROW

Society's ranking between two alternatives x and y should depend on how voters in the profile rank x and y... and nothing else.

AXIOM (INDEPENDENCE OF IRRELEVANT ALTERNATIVES, OR IIA)

A social welfare function F satisfies *Independence of Irrelevant Alternatives (IIA)* if, for any alternatives x and y and profiles $\boldsymbol{R}, \boldsymbol{R}'$ such that for any agent $i \in N$ it holds that $x \succ_i y$ if and only if $x \succ'_i y$, then it holds that $x \succ_{F(\boldsymbol{R})} y$ if and only if $x \succ_{F(\boldsymbol{R}')} y$.

If voters rank *x* and *y* in the same way across the two profiles, then the final ranking between *x* and *y* is the same for both profiles.



KENNETH ARROW

Non-dictatorship is about making sure that there is no one voter who has final say, regardless of the preferences of the other voters.

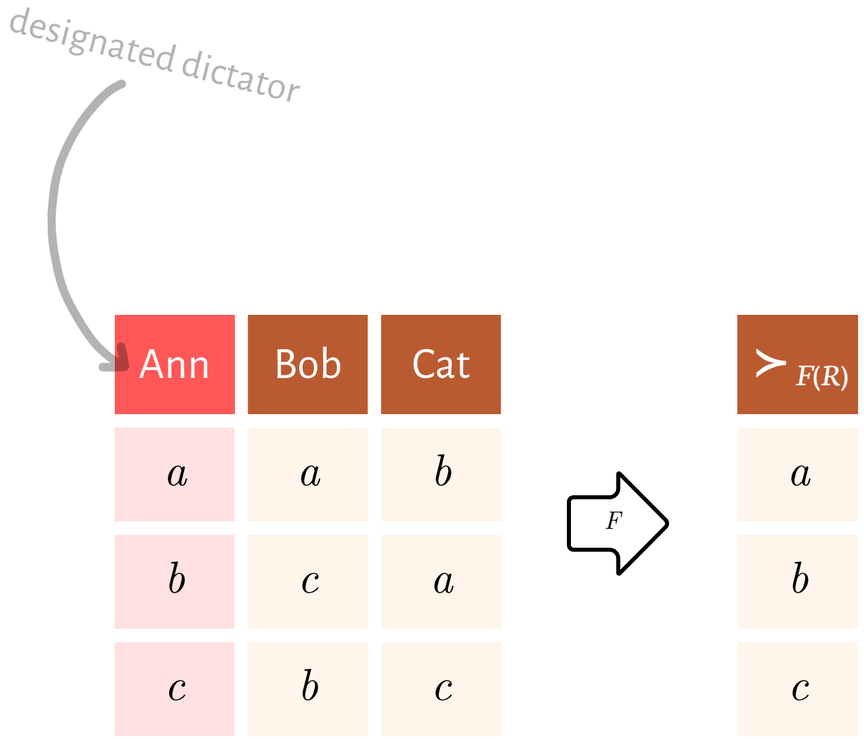
DEFINITION (DICTATOR)

An agent $i \in N$ is a dictator for a social welfare function F if, for any alternatives x and y and profile $\boldsymbol{R}$, it holds that if $x \succ_i y$, then $x \succ_{F(\boldsymbol{R})} y$.

AXIOM (NON-DICTATORSHIP)

A social welfare function F satisfies *Non-Dictatorship* if no agent is a dictator.

A dictator decides the final ranking of every pair of alternatives, and thus the full final ranking.



KENNETH ARROW

These properties seem reasonable enough. But it turns out that, together, they spell trouble.

THEOREM (ARROW, 1951)

If there are at least three alternatives, then any social welfare function that satisfies Pareto Efficiency and Independence of Irrelevant Alternatives is a dictatorial.

Arrow, K. (1951). *Social Choice and Individual Values*. John Wiley & Sons, Inc.

THEOREM (ARROW, 1951)

If there are at least three alternatives, then any social welfare function that satisfies Pareto Efficiency and Independence of Irrelevant Alternatives is a dictatorial.

PROOF (SKETCH)

The main steps of the proof:

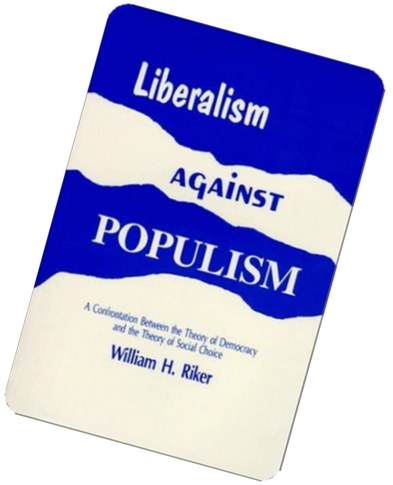
1. Extreme candidates end up in extreme positions.
2. For any alternative *z* there exists a voter *k* who is *pivotal for z*, i.e., can move *z* from the bottom of the aggregated ranking to the top, at some profile.
3. Pivotal voter *k* ends up being a dictator over any pair of alternatives *x* and *y* not involving *z*.
4. Voter k is a dictator over all pairs of alternatives.

Arrow, K. (1951). *Social Choice and Individual Values*. John Wiley & Sons, Inc.

WILLIAM H. RIKER

Arrow's theorem shows that democracy, as government of the will of the people, is an incoherent illusion.

There is no 'will of the people'!

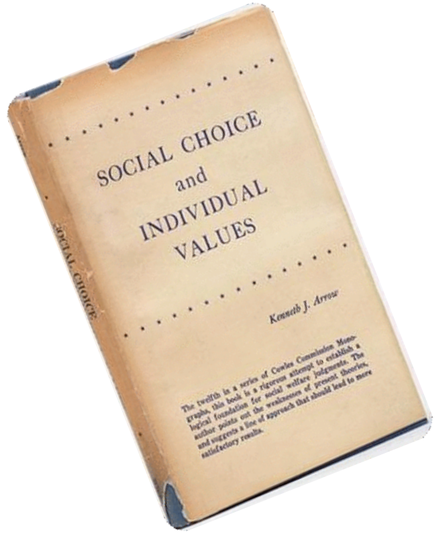


KENNETH ARROW

Certainly, it shows that certain intuitive, desirable properties are incompatible.

But then we have to lower our expectations.

It's all in the tradeoffs.



Riker, W.H. (1982). *Liberalism Against Populism: A Confrontation Between the Theory of Democracy and the Theory of Social Choice*. San Francisco: W.H. Freeman.
Arrow, K. (1951). *Social Choice and Individual Values*. John Wiley & Sons, Inc.
Morreau, M, (2019). Arrow's Theorem, Edward N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy (Winter 2019 Edition)*.

The Strategy of Voting

What better way to think of how voting can go awry than by looking at the decision practices of FIFA…

What voting rule is being used here?

	ROUND 1	ROUND 2	ROUND 3	ROUND 4
QATAR	11	10	11	14
USA	3	5	6	8
SOUTH KOREA	4	5	5	
JAPAN	3	2		
AUSTRALIA	1			

2018 and 2022 FIFA World Cup bids. Wikipedia

What voting rule is being used here?

Some version of Instant-Runoff Voting, or STV.

Where voters submit a new vote at every round.

What is going on with the votes for Qatar between rounds 1 and 2 though?

	ROUND 1	ROUND 2	ROUND 3	ROUND 4
QATAR	11	10	11	14
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- Where voters submit a new vote at every round.

What is going on with the votes for Qatar between rounds 1 and 2 though?

- Perhaps one of the voters for Qatar, anticipating a tie-break between the US and Japan in round 2, casts their vote for the US instead in order to ensure its survival to round 3.
- Similarly, it seems that one supporter of Japan suspects that Japan will be kicked out at round 2 and goes for their second-best?

	ROUND 1	ROUND 2	ROUND 3	ROUND 4
QATAR	11	10	11	14
USA	3	5	6	8
SOUTH KOREA	4	5	5	
JAPAN	3	2		
AUSTRALIA	1			

Either way, the changing tallies suggest tactical voting.

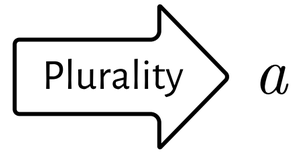
2018 and 2022 FIFA World Cup bids. Wikipedia

We've seen that many voting rules are afflicted by a common problem: they create incentives for voters to lie about their preferences.

Recall…

Under Plurality, voters don't want to support a losing candidate.

49	48	3
a	b	c
b	c	b
c	a	a



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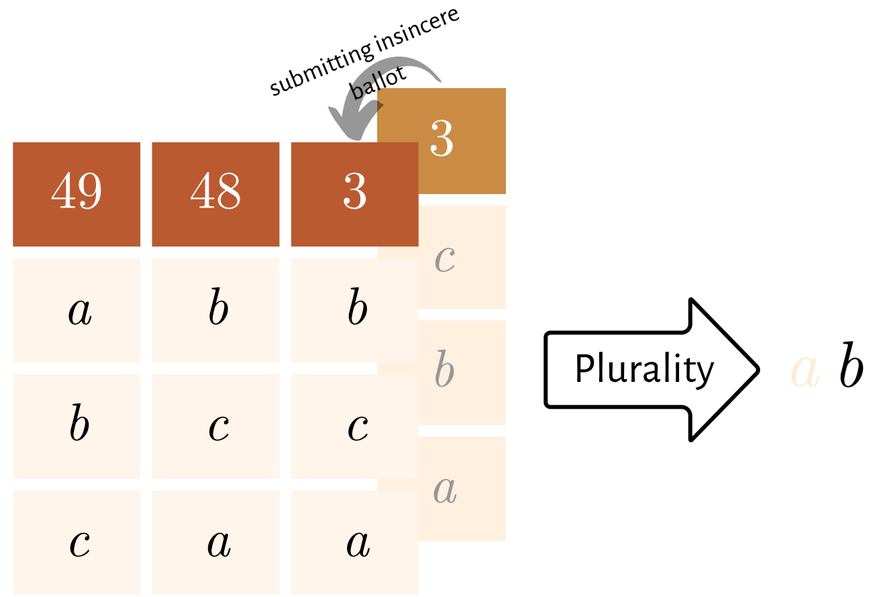
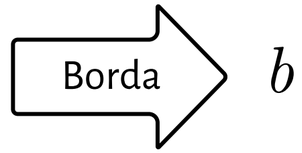


Under Borda, voters can manipulate by pushing alternatives they don't like down their list.

2	1
b	a
a	b
c	c
d	d



b

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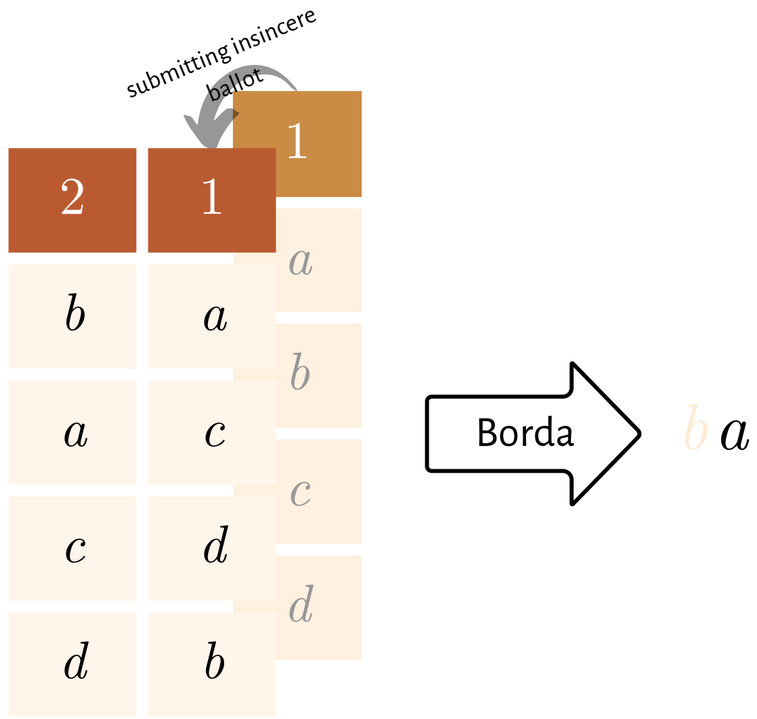

BORDA

Hey! I said honest people!

CHARLES DODGSON

Strategizing makes an election more of a game of skill than a real test of the wishes of the electors.

DEFINITION (STRATEGYPROOFNESS)

A resolute social choice function F is *strategyproof* if for all voters $i \in N$ it holds that there does not exist a profile $\boldsymbol{R}$ and some order $\succ'_i$ such that:

$$F(\succ'_i, \boldsymbol{R}_{-i}) \succ_i F(\boldsymbol{R}).$$

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$$F(\succ'_i, \boldsymbol{R}_{-i}) \succ_i F(\boldsymbol{R}).$$

insincere preference

not essential that the voting rule is resolute; but if it isn't we need to clarify what it means for one outcome to be preferred to another

containing i's true preference

profile without i

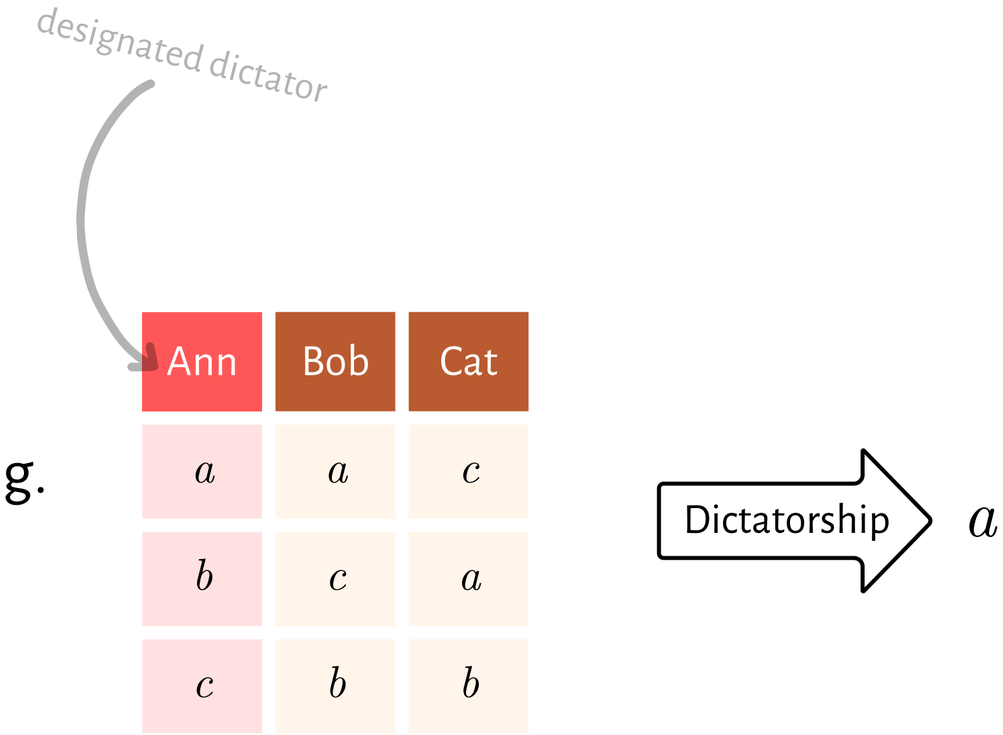
profile with i's truthful preference

Can we design strategyproof voting rules?

DEFINITION (DICTATORSHIP)

Choose an agent $i \in N$, called *the dictator*. The winner is the top choice of the dictator.

Under Dictatorship there is no point in manipulating.



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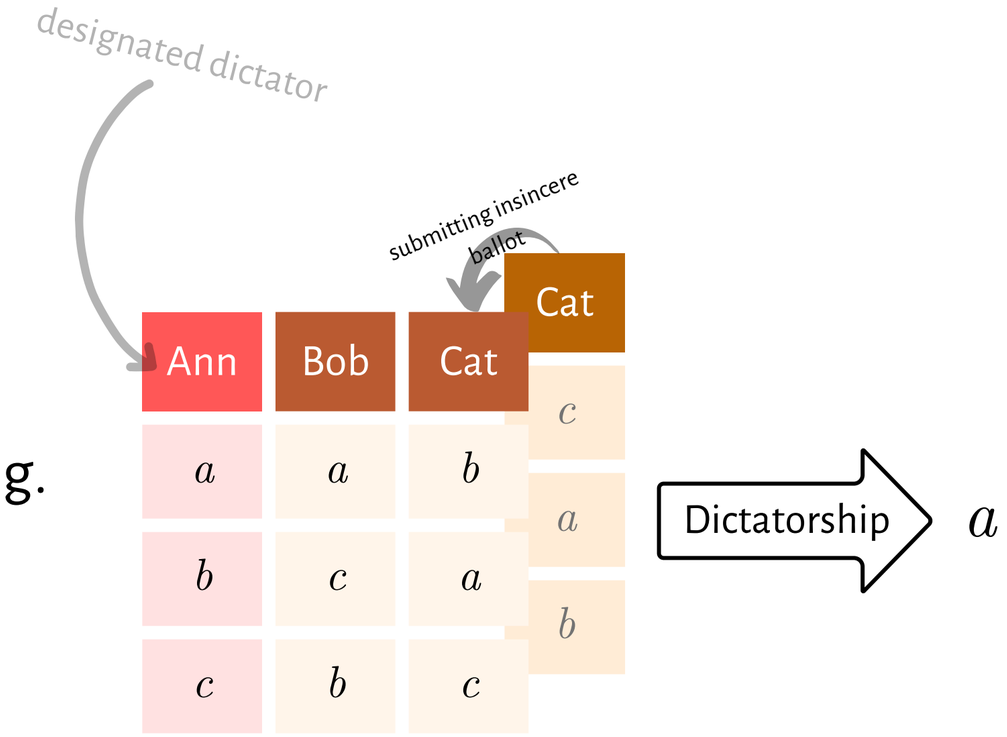
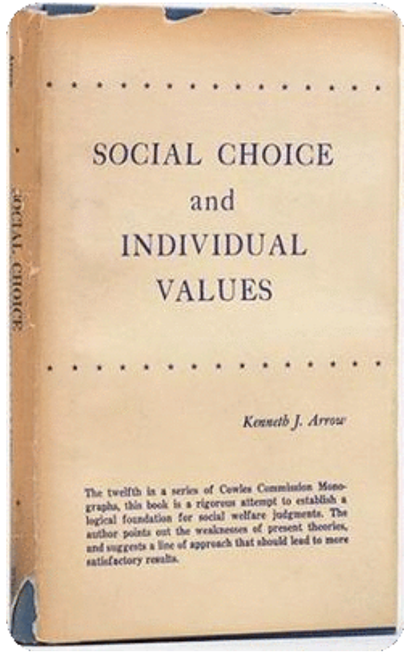


KENNETH ARROW

Recall that dictatorship is the only rule that satisfies Pareto Efficiency and Independence of Irrelevant Alternatives.



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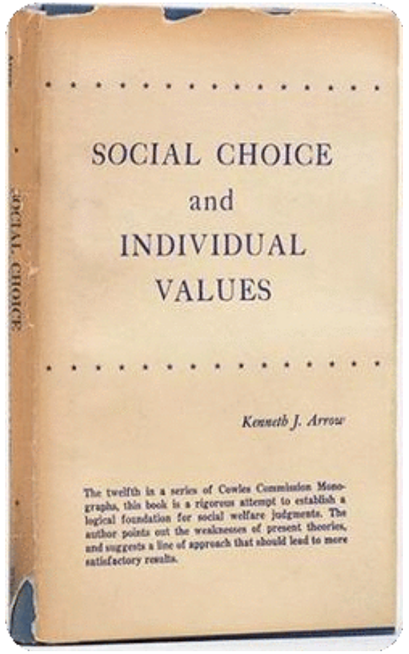
KIM JONG UN

I like this result.



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ALLAN GIBBARD

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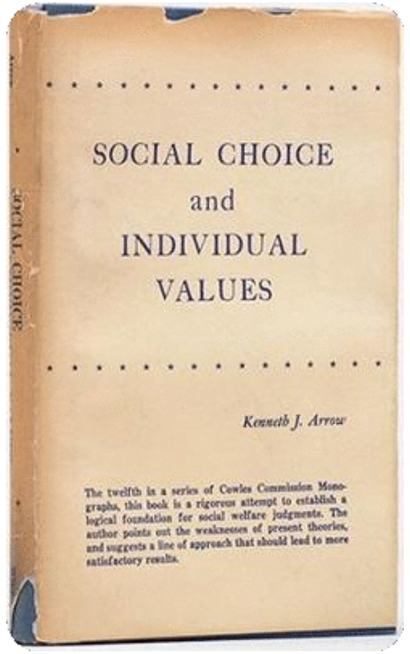

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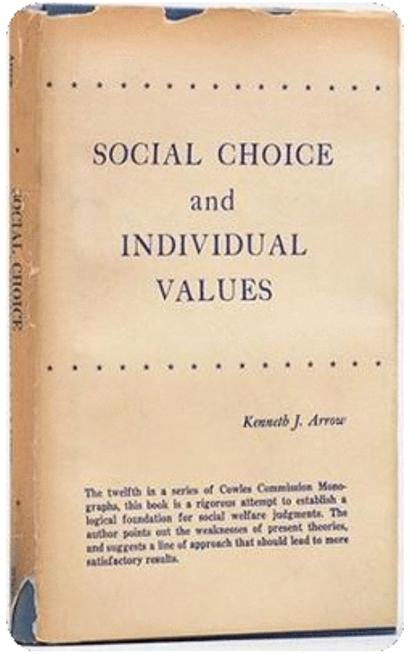
MARK SATTERTHWAITE

Yeah about that…



Arrow, K. (1951). *Social Choice and Individual Values*. John Wiley & Sons, Inc.

THEOREM (GIBBARD-SATTERTHWAITE, 1973, 1975)

If a resolute social choice function F has at least three possible outcomes, then F is strategyproof if and only if it is a dictatorship.

Gibbard, A. (1973). Manipulation of voting schemes: A general result. *Econometrica*. 41 (4): 587–601.
Satterthwaite, M. (1975). Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*. 10 (2): 187–217.

DUNCAN BLACK

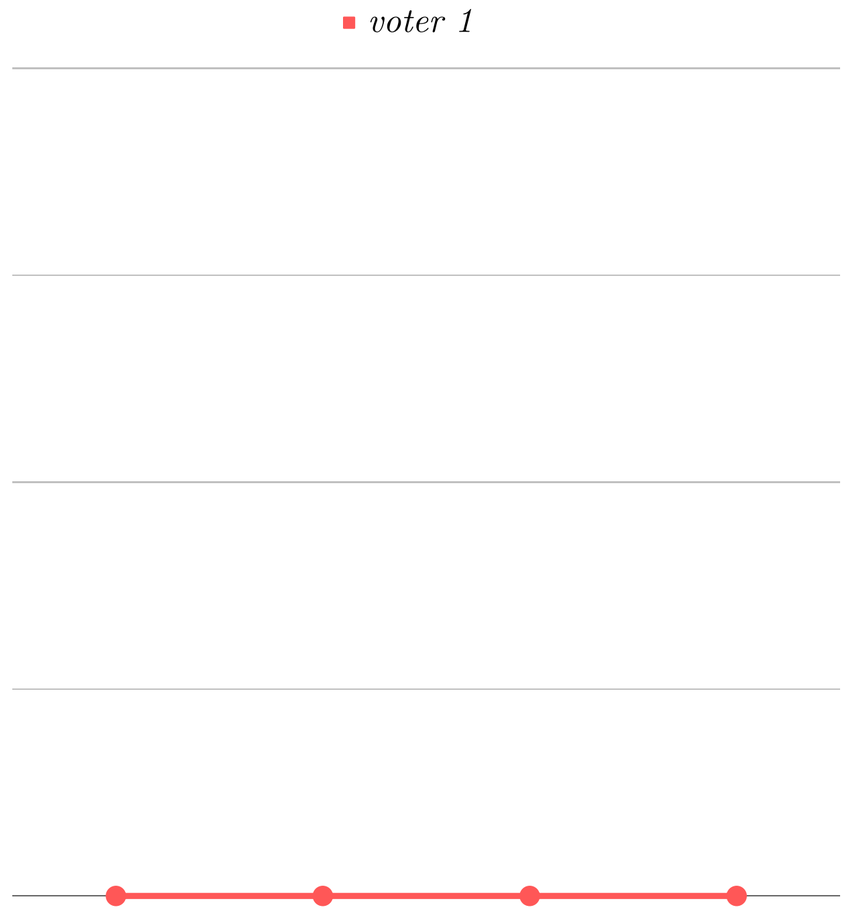
Another way of escaping impossibility results is to assume the input, i.e., preference profiles, have some more specific structure.

DEFINITION (SINGLE-PEAKED PROFILES)

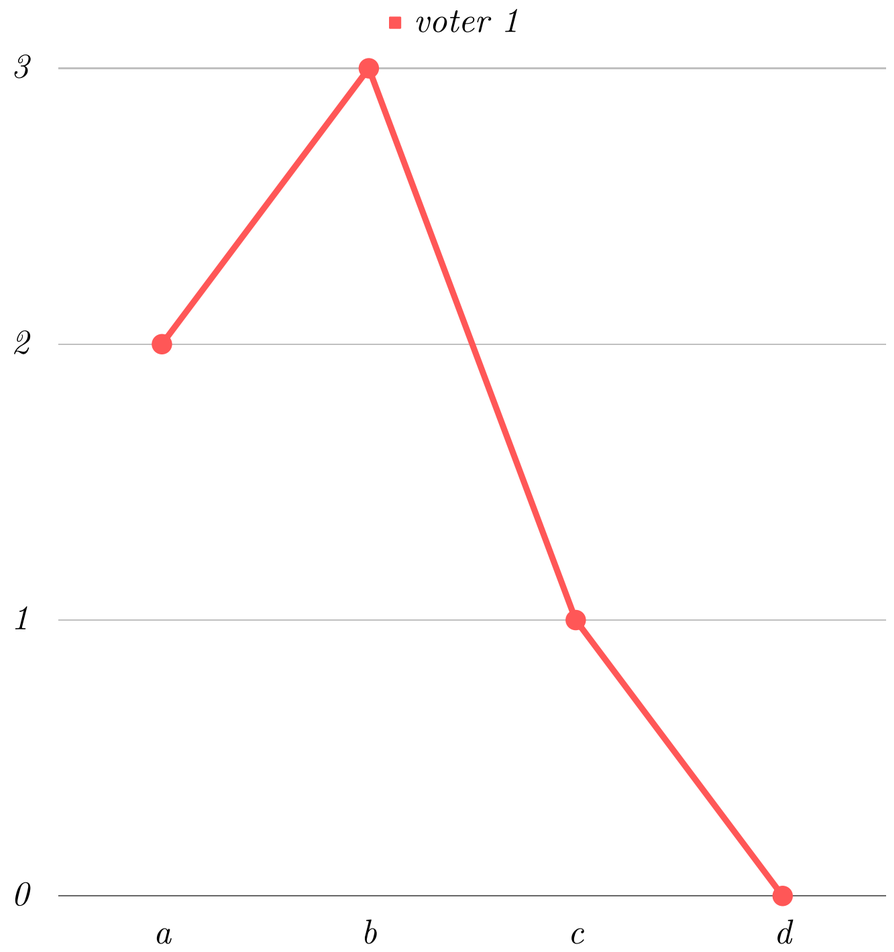
A profile is *single-peaked* if :

1. alternatives can be ordered linearly, e.g., from left to right, and
2. every voter has a most preferred alternative, and other alternatives are less preferred the further away they are to the ideal one.

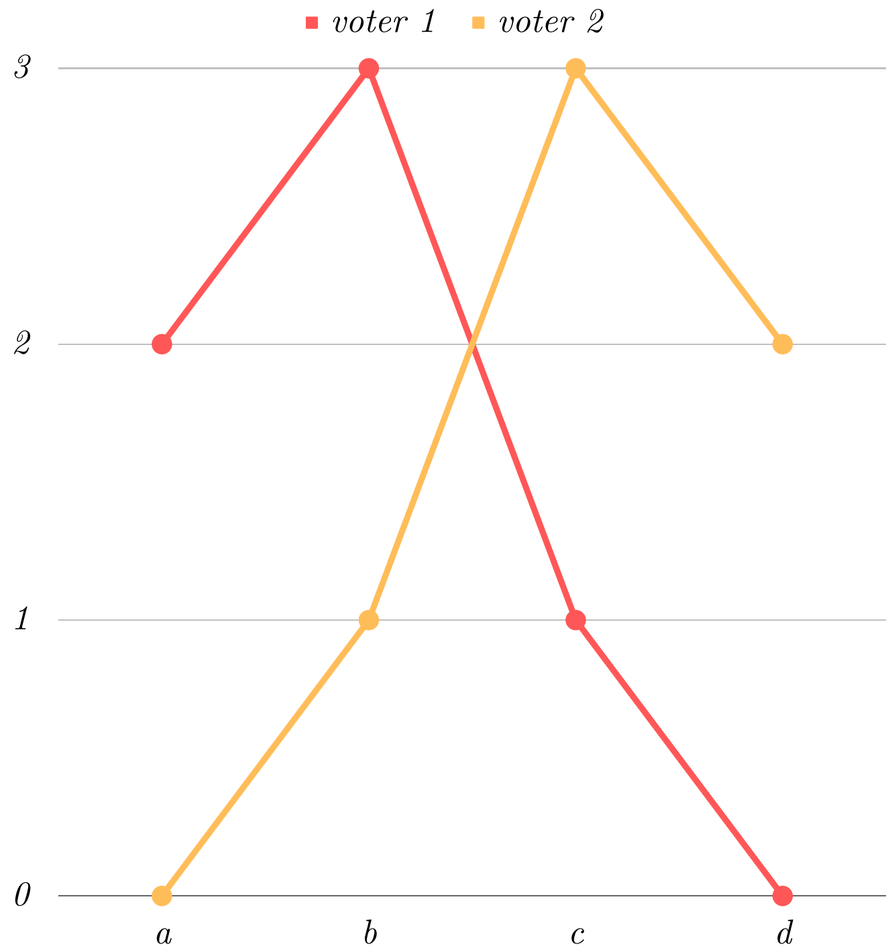
$\succ_1$	$\succ_2$	$\succ_3$
b	c	a
a	d	b
c	b	c
d	a	d



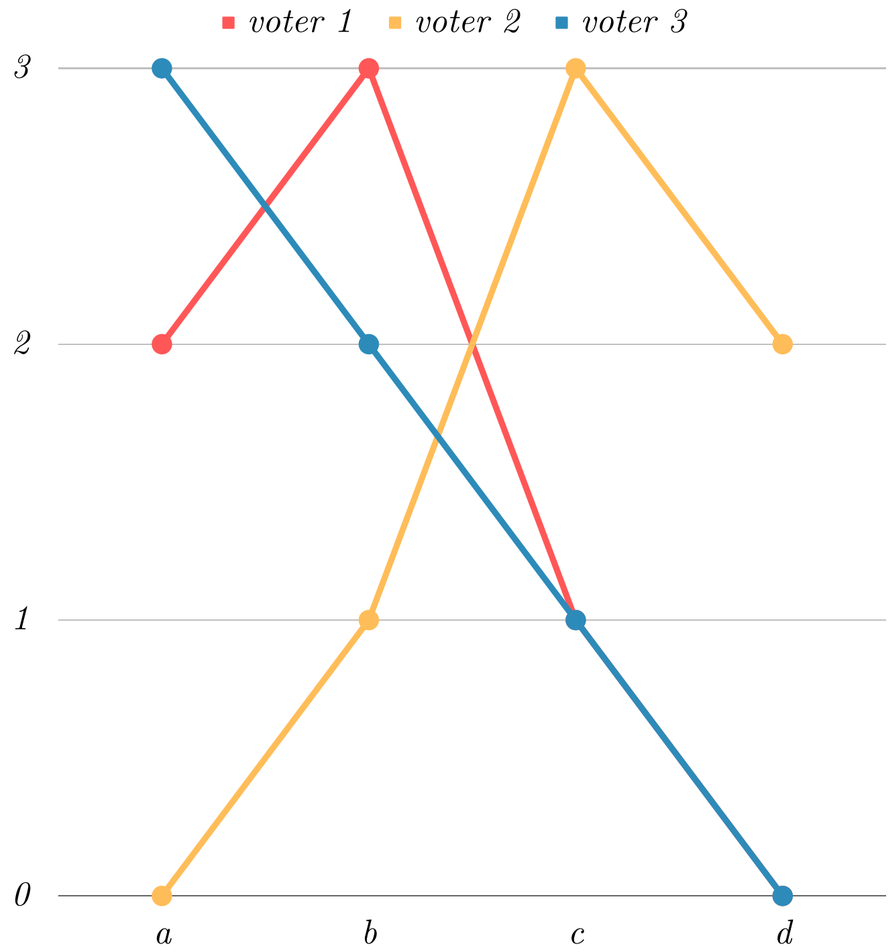
$\succ_1$	$\succ_2$	$\succ_3$
b	c	a
a	d	b
c	b	c
d	a	d



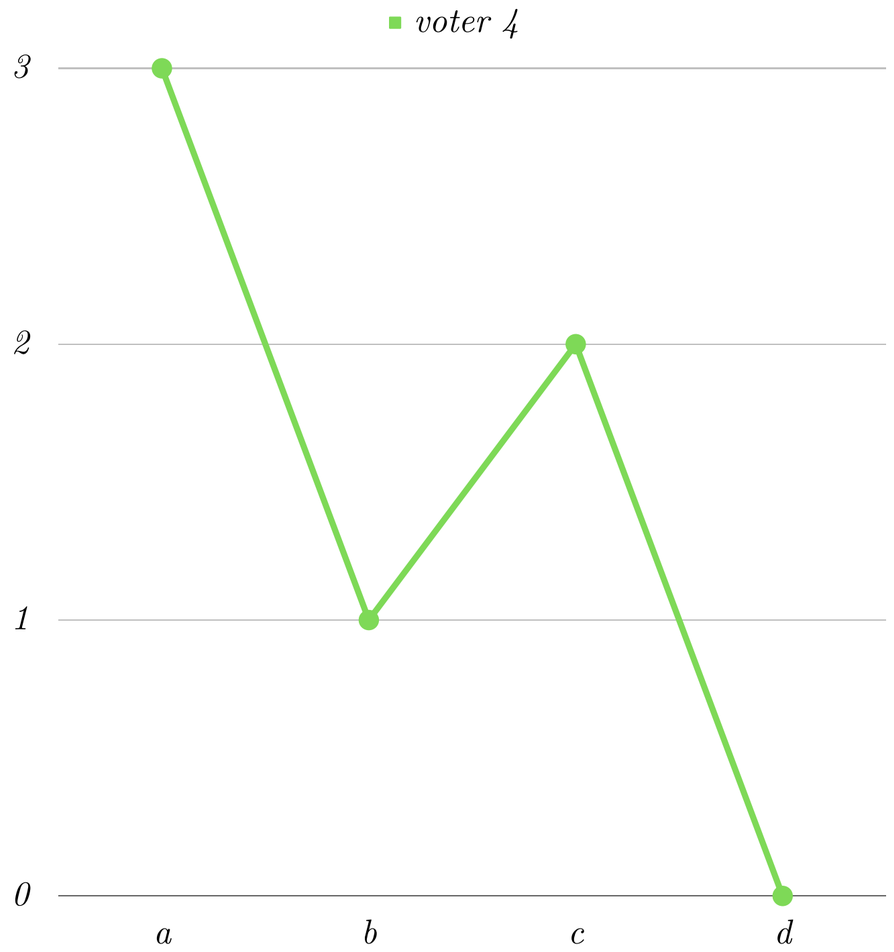
$\succ_1$	$\succ_2$	$\succ_3$
b	c	a
a	d	b
c	b	c
d	a	d



$\succ_1$	$\succ_2$	$\succ_3$
b	c	a
a	d	b
c	b	c
d	a	d

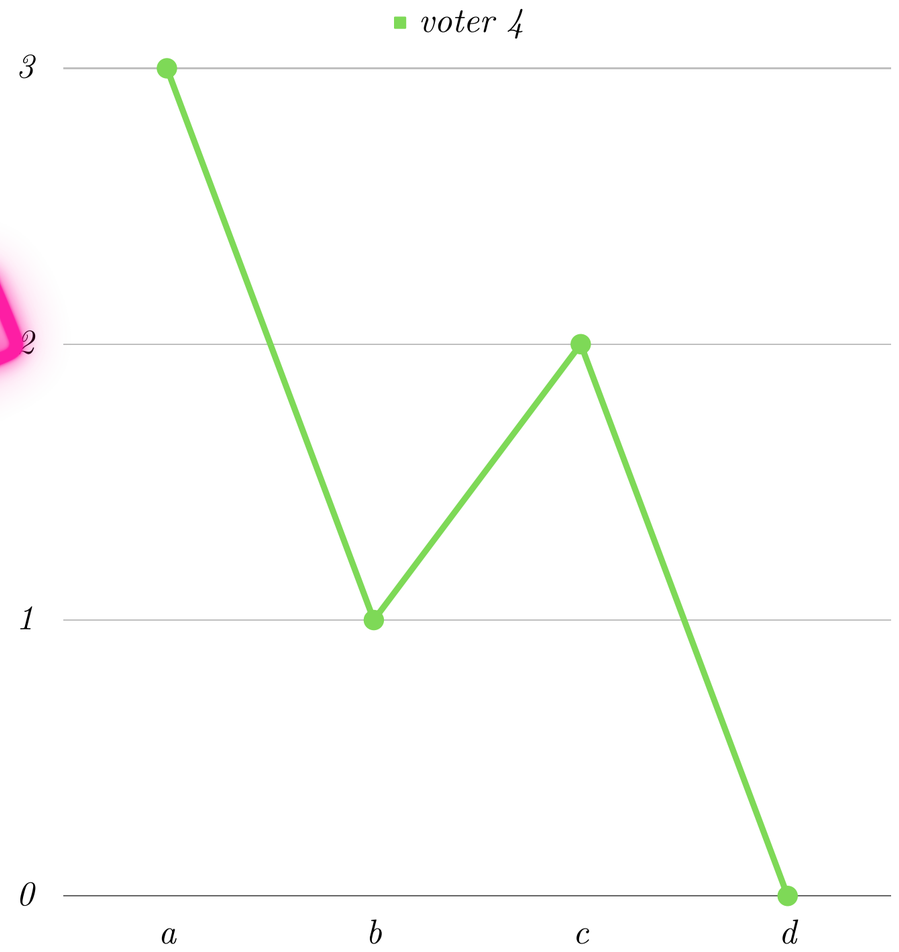


$\succ_4$
a
c
b
d



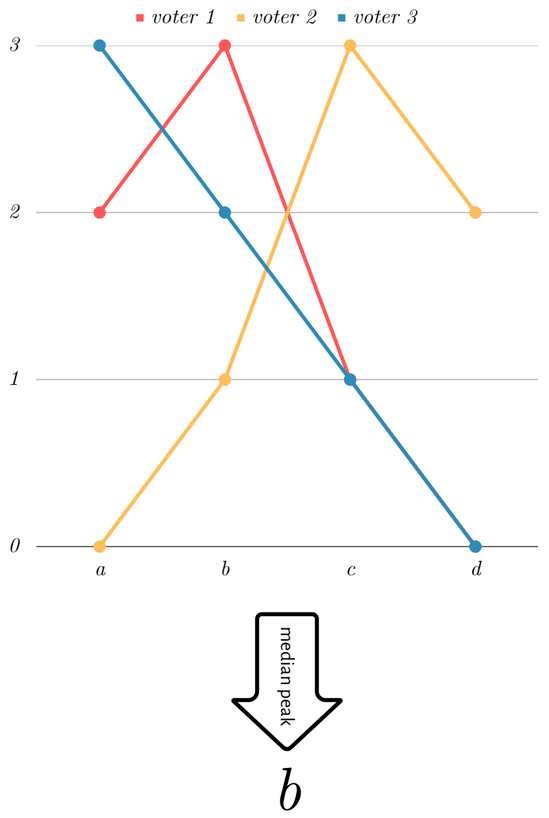
$\succ_4$
a
c
b
d

NOT SINGLE-PEAKED



THEOREM (BLACK, 1948)

For an odd number of voters, if the profile is single-peaked then the median peak is a Condorcet winner and the Condorcet rule is strategyproof.



Black, D. (1948). On the Rationale of Group Decision-making. *Journal of Political Economy*, 56: 23-34.

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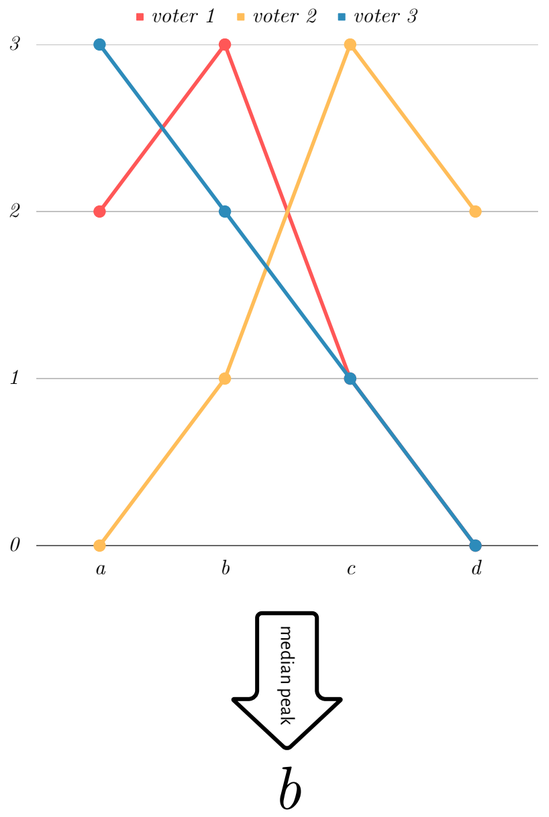
For an odd number of voters, if the profile is single-peaked then the median peak is a Condorcet winner and the Condorcet rule is strategyproof.

PROOF

If alternative x^* is the median peak, all voters whose peak is to the right of (or including) x^* rank x^* higher than alternatives to its left. And there is a strict majority of such voters. Similarly, voters whose peak is the left of (or including) x^* rank x^* higher than alternatives to its left. Thus, x^* beats every other alternative in a head-to-head contest, i.e., is a Condorcet winner.



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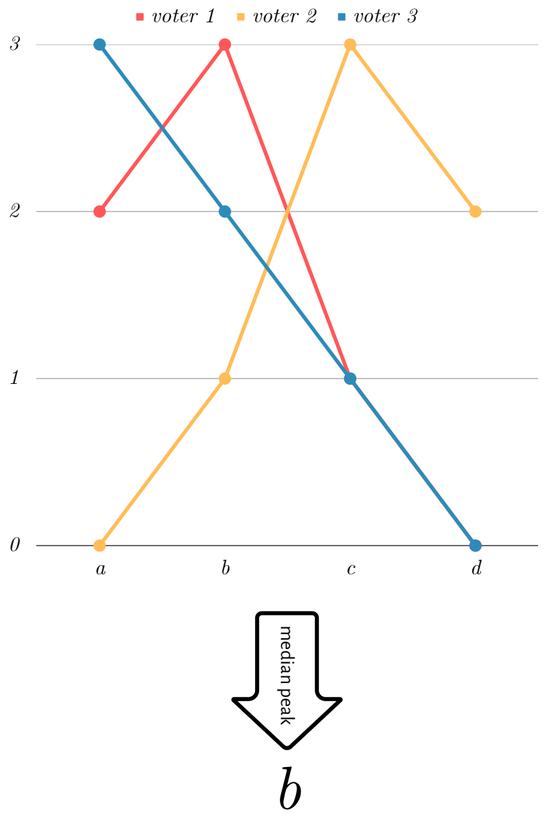
For strategyproofness, note that the median voter does not want to change their vote, and for every other voter a change of the result leads to a worse alternative being elected.



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THEOREM

Approval voting is strategyproof.

THEOREM

Approval voting is strategyproof.

PROOF

If an alternative you approve of is the winner, then there is no gain from being insincere.

If the winner is an alternative you do not approve of, then there is no way of supporting your approved alternatives other than putting them on your ballot.