LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

RESEARCH SEMINAR IN DECISION AND ACTION THEORY

The Wisdom of Interacting Crowds

Adrian Haret

Nicolien Janssens

Giuseppe Dari Mattiacci

Frederik Van De Putte

Davide Grossi

OCTOBER 25, 2023

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Deliberating?

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Sometimes groups get it right.

Does the city of Munich have more than 1.5 million inhabitants?

- ☐ no
- ☐ yes

Does the city of Munich have more than 1.5 million inhabitants?

- ☐ no
- ☑ yes

Sometimes groups get it wrong.

ODORIC OF PORDENONE

In a province of the Grand Can there grow gourds, which, when they are ripe, open, and within them is found a little beast like unto a young lamb…

Odoric of Pordenone [trans. Sir Henry Yule] (2002). *The Travels of Friar Odoric*. W.B. Eerdmans Publishing Company.

AD 1330

SIR JOHN MANDEVILLE

In Tartary groweth a manner of fruit, as though it were gourds. And when they be ripe, men cut them a-two, and men find within a little beast, in flesh, in bone, and blood, as though it were a little lamb without wool. And men eat both the fruit and the beast. And that is a great marvel.

Of that fruit I have eaten...

and found it wondirfulle.

Mandeville, J. (1900). *The Travels of Sir John Mandeville. The Cotton Manuscript in modern spelling.* Macmillan and Co. Limited.

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BARON SIGISMUND VON HERBERSTEIN

AD 1515 - 1553

[...] a certain seed like that of a melon, but rather rounder and longer, from which, when it was set in the earth, grew a plant resembling a lamb, and attaining to a height of about two and a half feet...

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ATHANASIUS KIRCHER

[...] we assert that it is a plant. Though its form be that of a quadruped, and the juice beneath its woolly covering be blood which flows if an incision be made in its flesh, these things will not move us. It will be found to be a plant.

Kircher, A. (1641). *Magnes; sive de arte magneticâ opus tripartitum.*

AD 1641

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AD 1641

ENGELBERT KAEMPFER

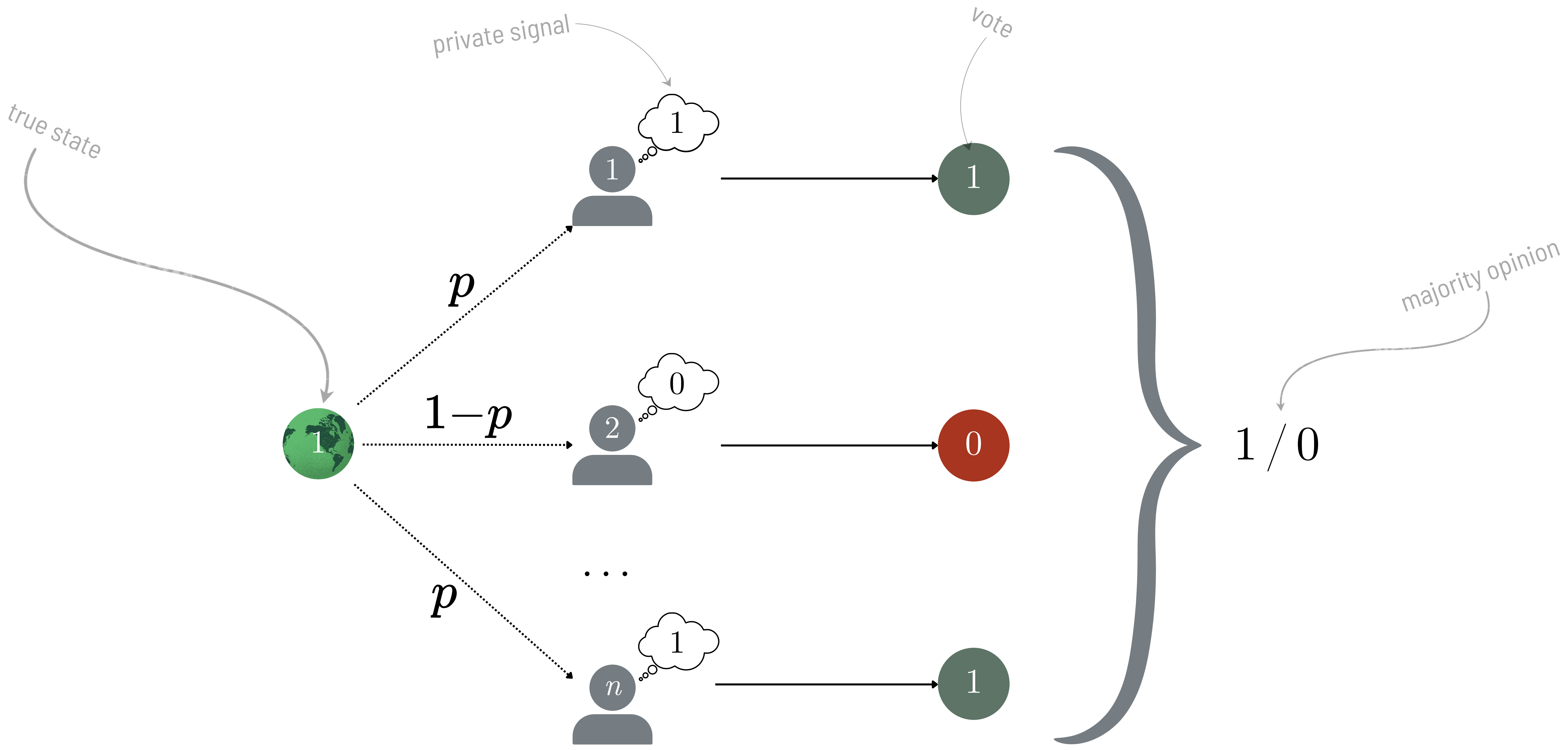
I have searched ad risum et nauseam for this zoophyte feeding on grass, but have found nothing.

Kaempfer, E. (1712). *Amœnitatum Exoticarum politico-physico-medicarum fascicul.*

AD 1683

How to think of opinion formation?

Agents as Noisy Estimators of the Truth



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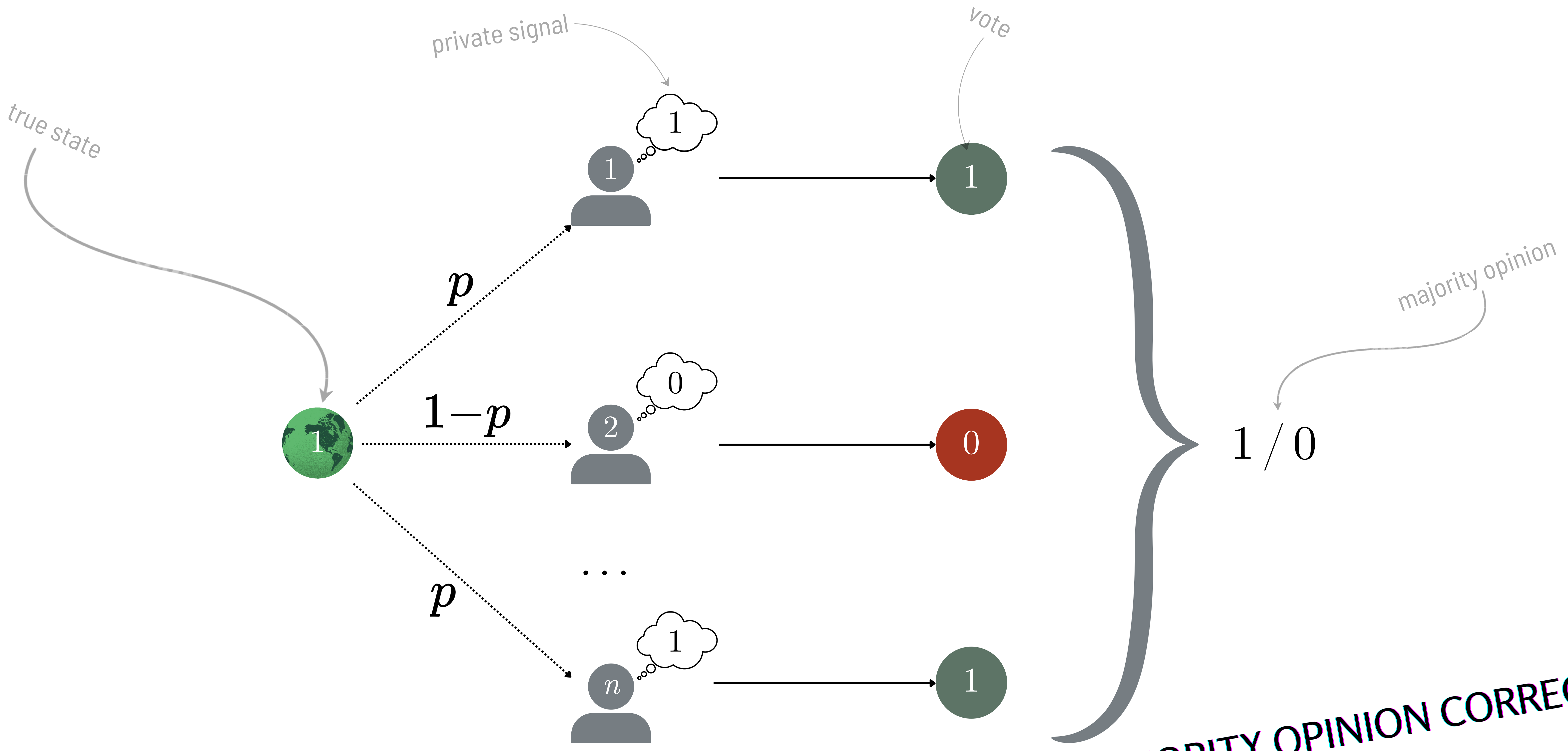


WHEN IS THE MAJORITY OPINION CORRECT?

CONDORCET

I *claim that the majority will be correct!*

CONDORCET

I *claim that the majority will be correct!*

Most of the time...

CONDORCET

I claim that the majority will be correct!

Most of the time…

Under some conditions…

Assumptions

COMPETENCE

Agents are *competent*, i.e., better than random at being correct:

probability of vote for correct alternative

$$p > \frac{1}{2}, \text{ for every agent.}$$

INDEPENDENCE

Agents vote *independently* of each other:

$$\Pr\left[i \text{ votes } x, j \text{ votes } y\right] = \Pr\left[i \text{ votes } x\right] \cdot \Pr\left[j \text{ votes } y\right], \text{ for any two agents } i \text{ and } j.$$

THEOREM (THE CONDORCET JURY THEOREM)

Under the previous assumptions, it holds that:

$$\Pr\Big[\text{majority of } n+2 \text{ are correct}\Big] > \Pr\Big[\text{majority of } n \text{ are correct}\Big]$$

and

$$\lim_{n\to\infty} \Pr\Big[\text{majority of } n \text{ are correct}\Big] = 1$$

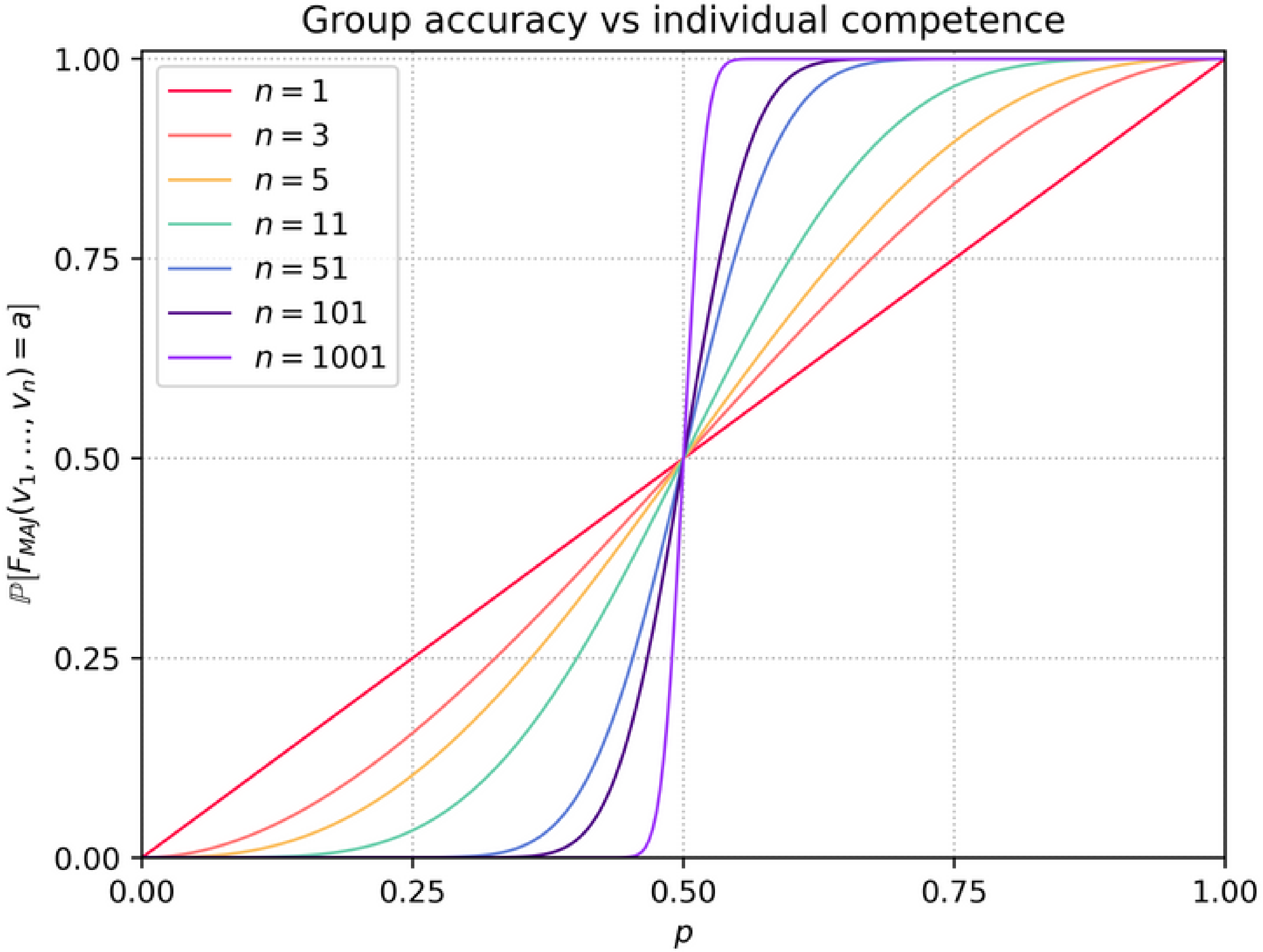
assumed odd

probability of correct decision grows with the size of the group

in the limit, majority opinion is perfect

CONDORCET

And behold! Group accuracy grows quickly with individual accuracy.



NICOLIEN

What if people talk & persuade each other?

CONDORCET

NICOLIEN

What if people talk & persuade each other?

CONDORCET

Doesn't sound like a good idea to me.

NICOLIEN

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Doesn't sound like a good idea to me.

HÉLÈNE LANDEMORE

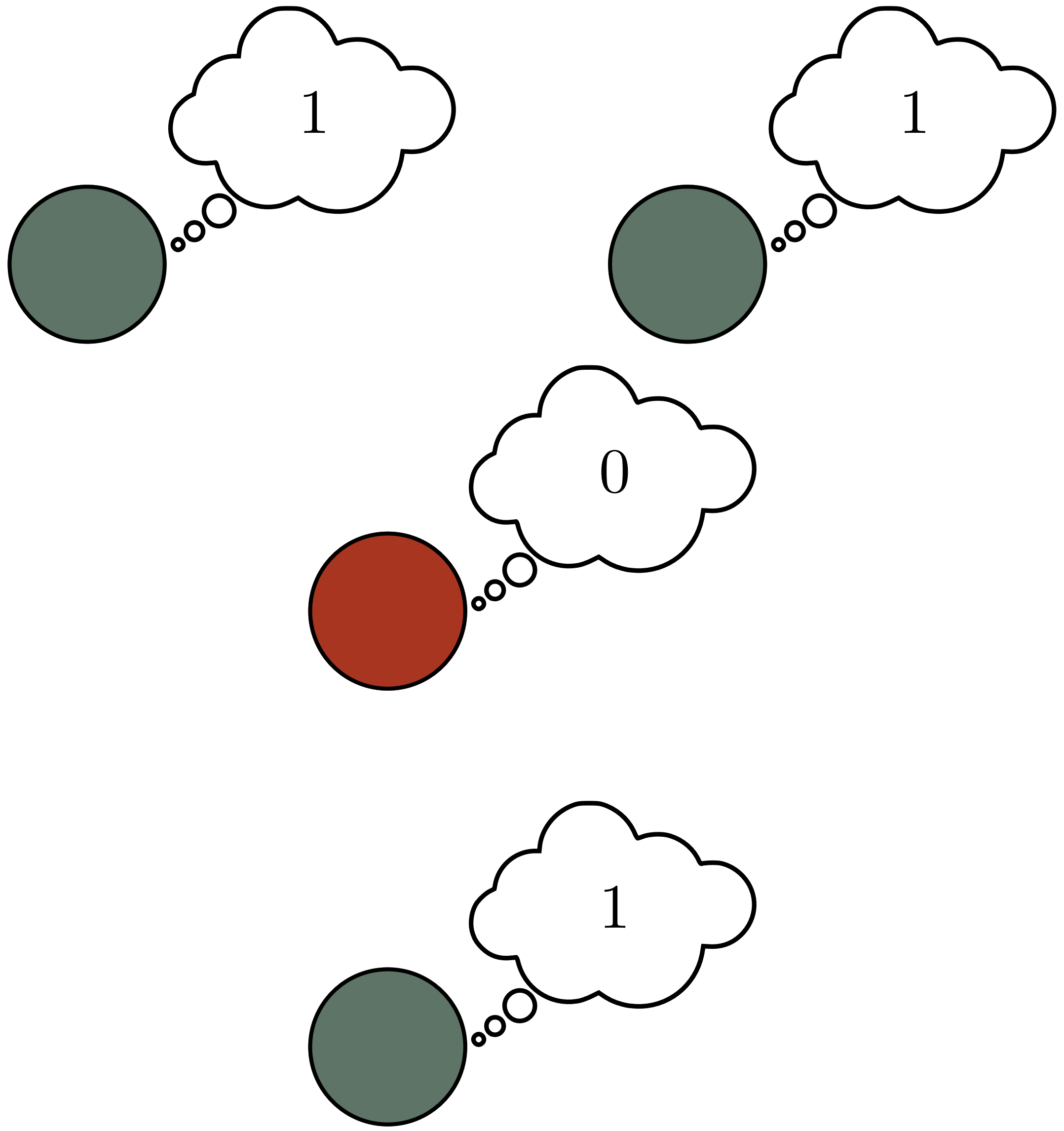
The first, most obvious, and perhaps oldest mechanism that makes democracy an epistemically reliable decision procedure is deliberation.

Landemore, H. (2013). *Democratic Reason: Politics, Collective Intelligence, and the Rule of the Many.* Princeton University Press.

Deliberation?

Enter ~~Communication~~

We start with the standard Condorcet Jury Theorem setup, and add the assumption that agents are in a social network.



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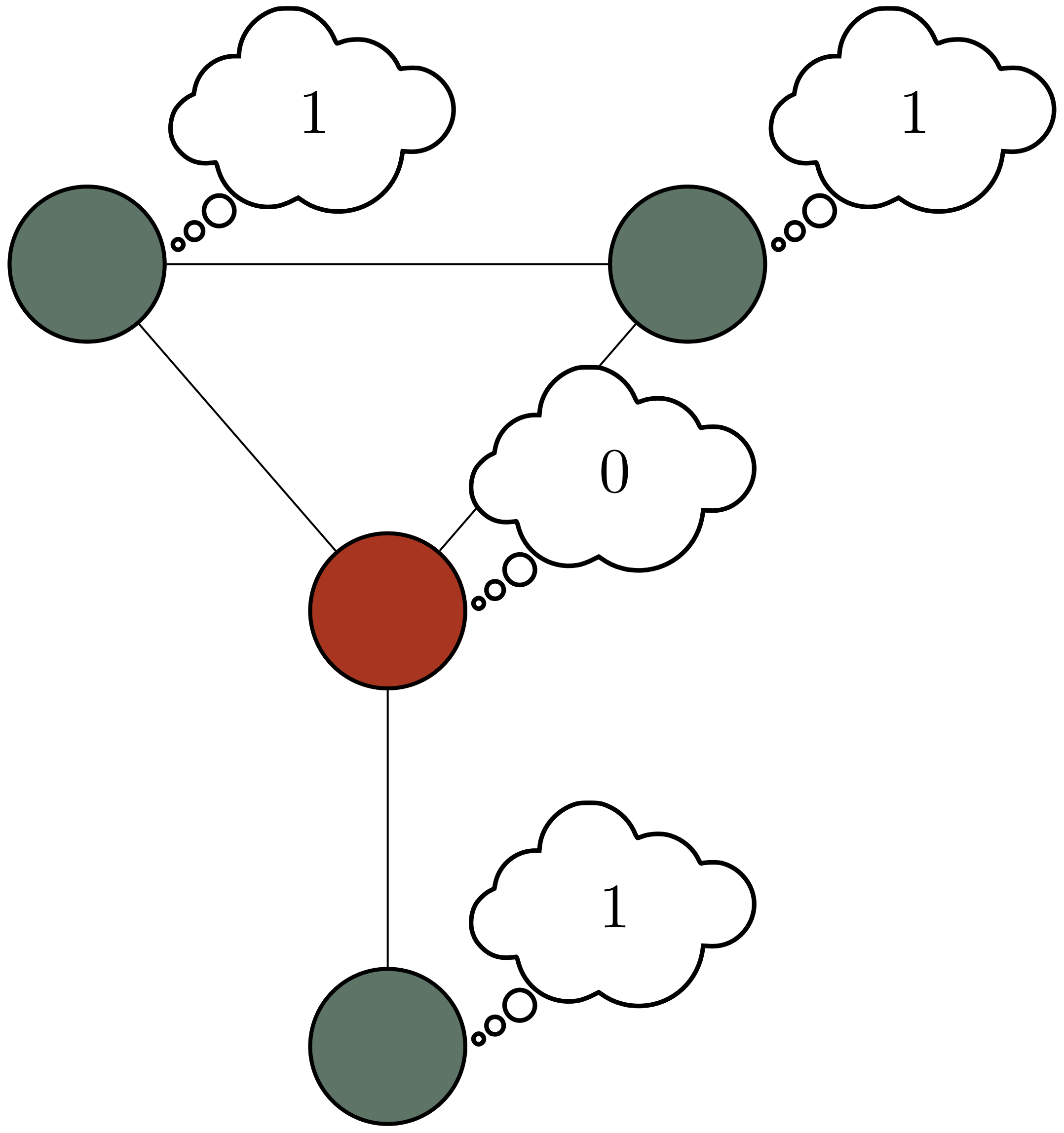
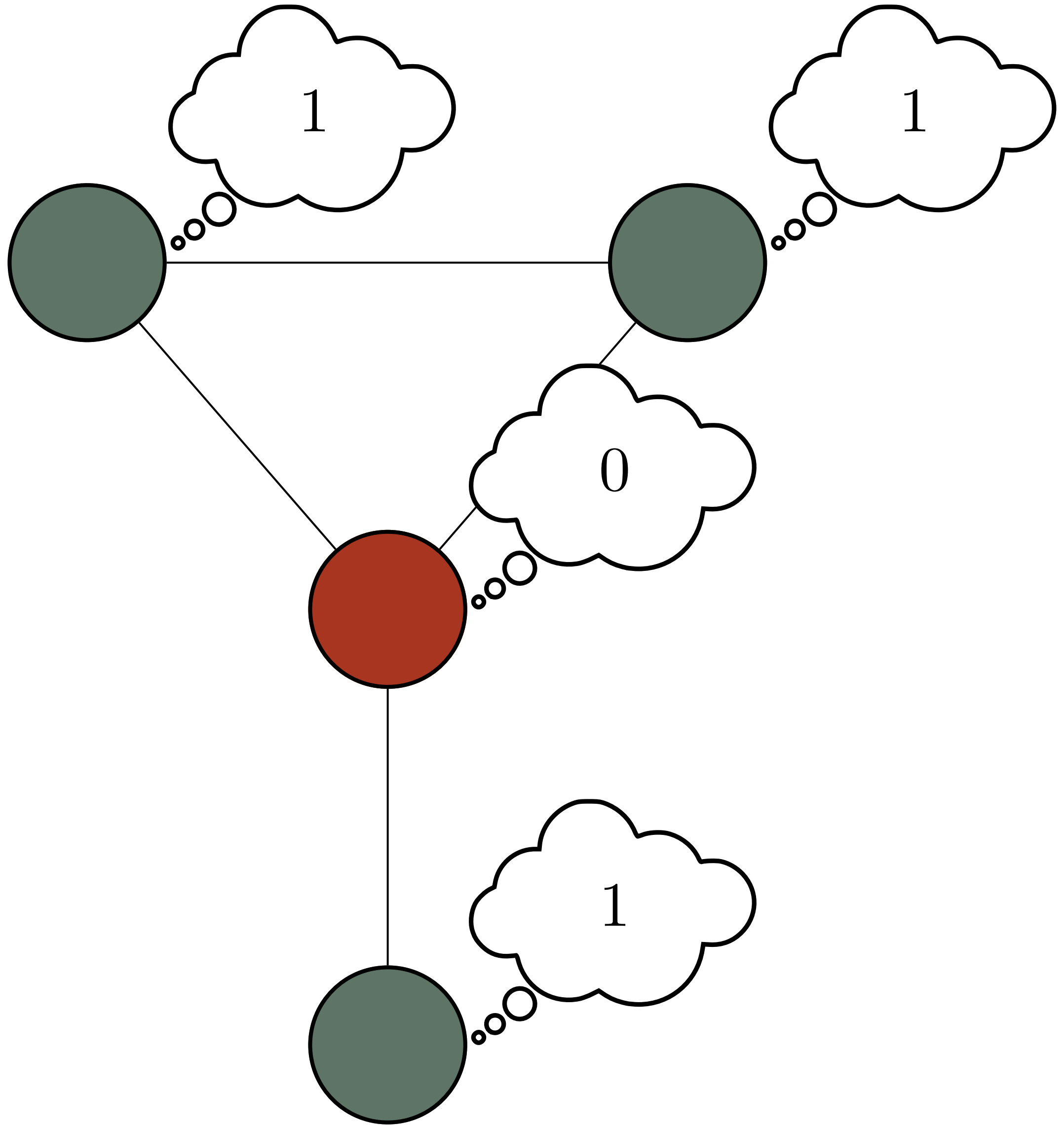


NICOLIEN

We also add a deliberation phase, in which agents share their (independent) private signals with their neighbors.



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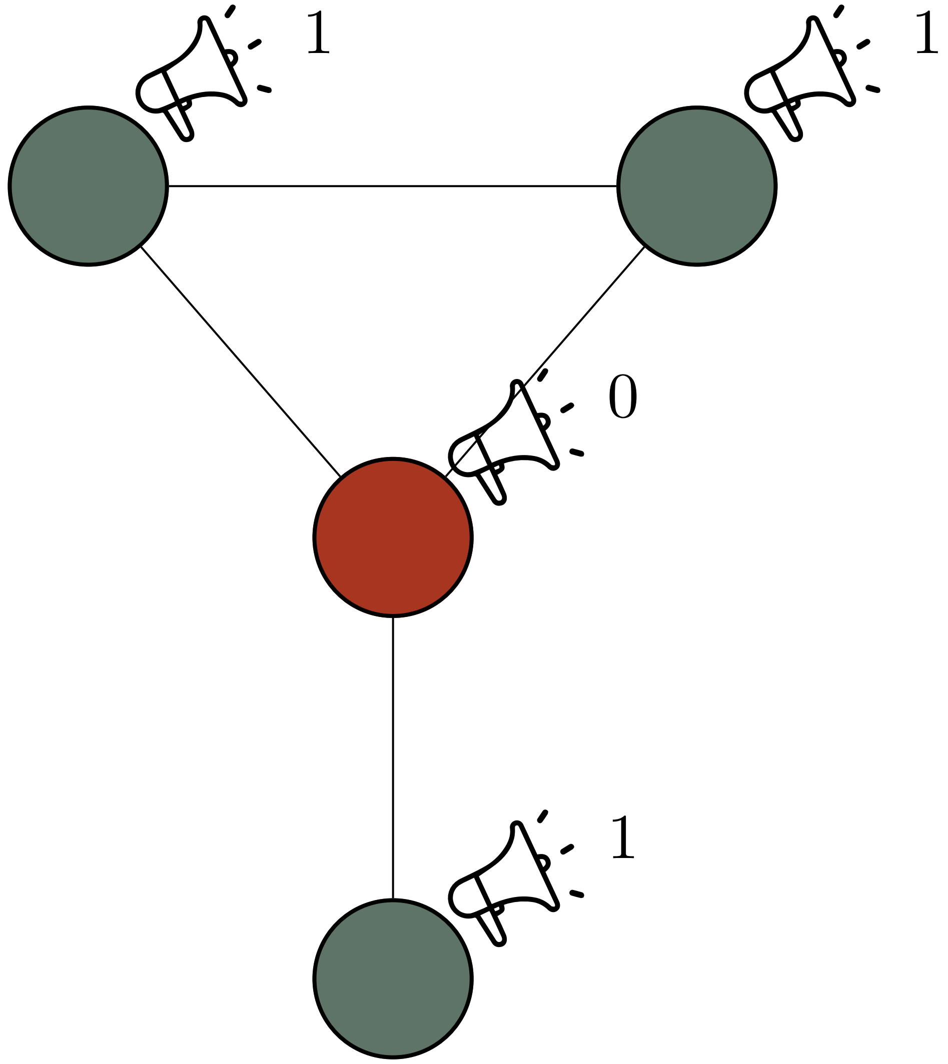

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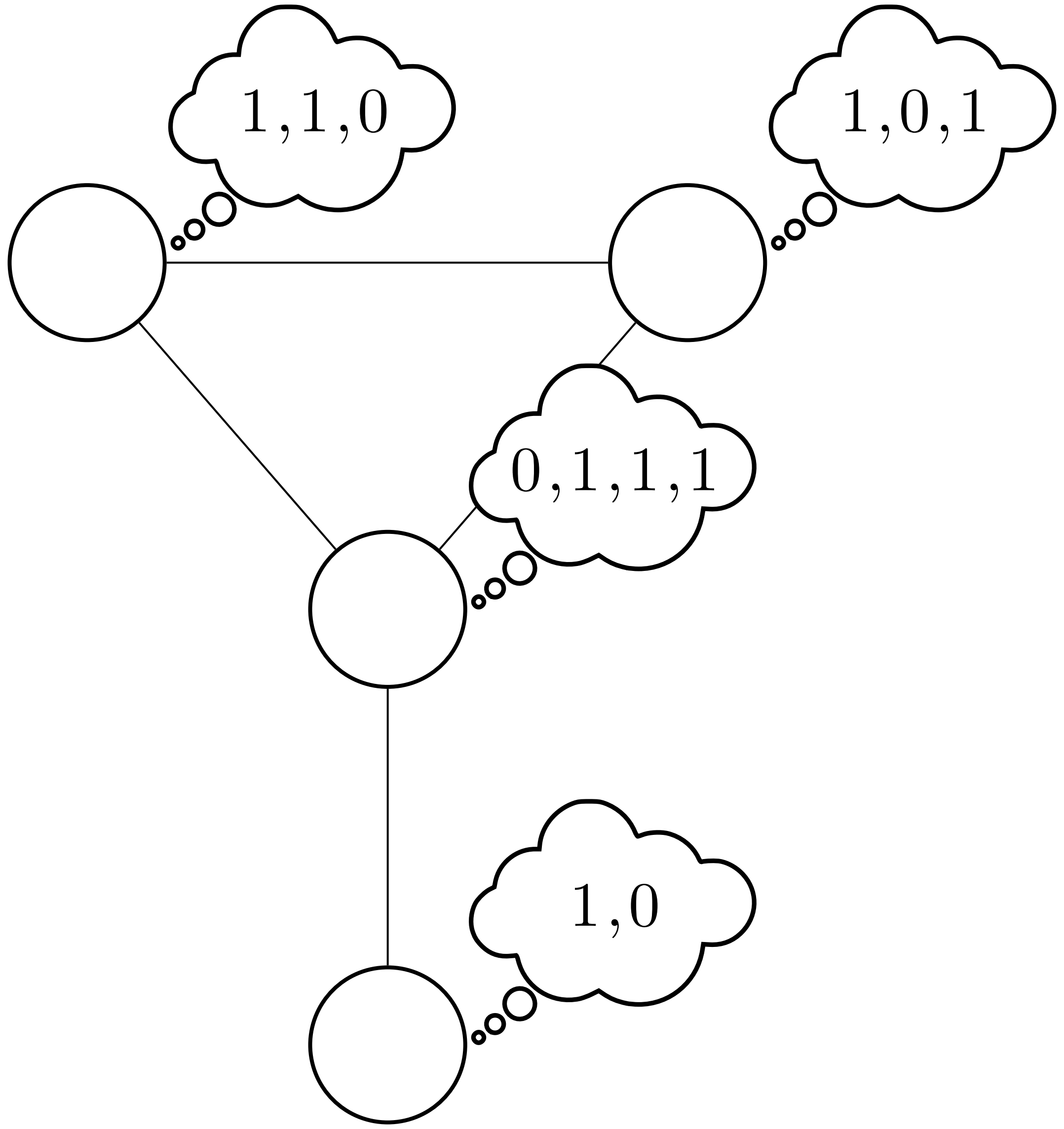

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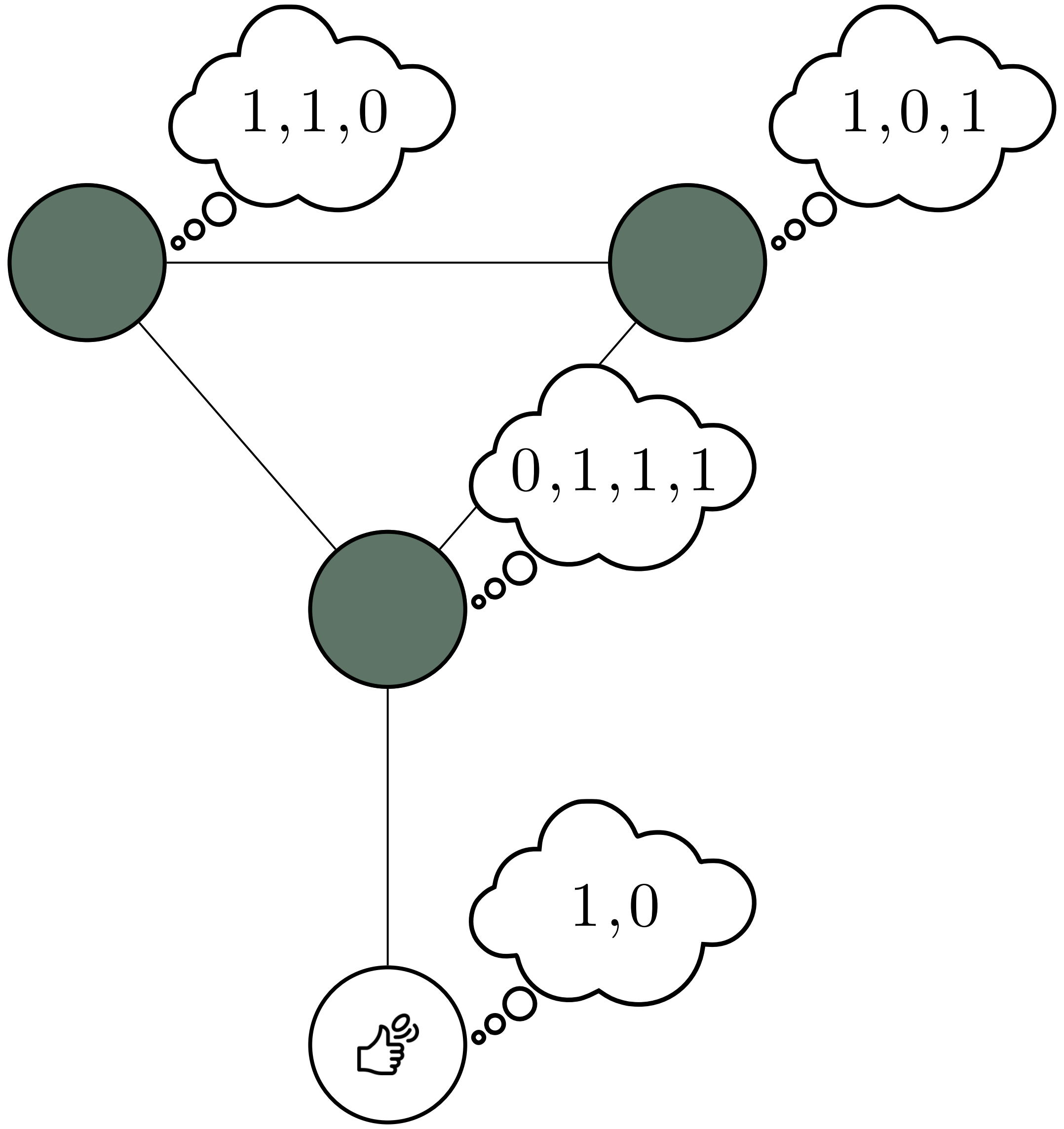
ADRIAN

After which agents update their opinions: to the majority of the signals they see.



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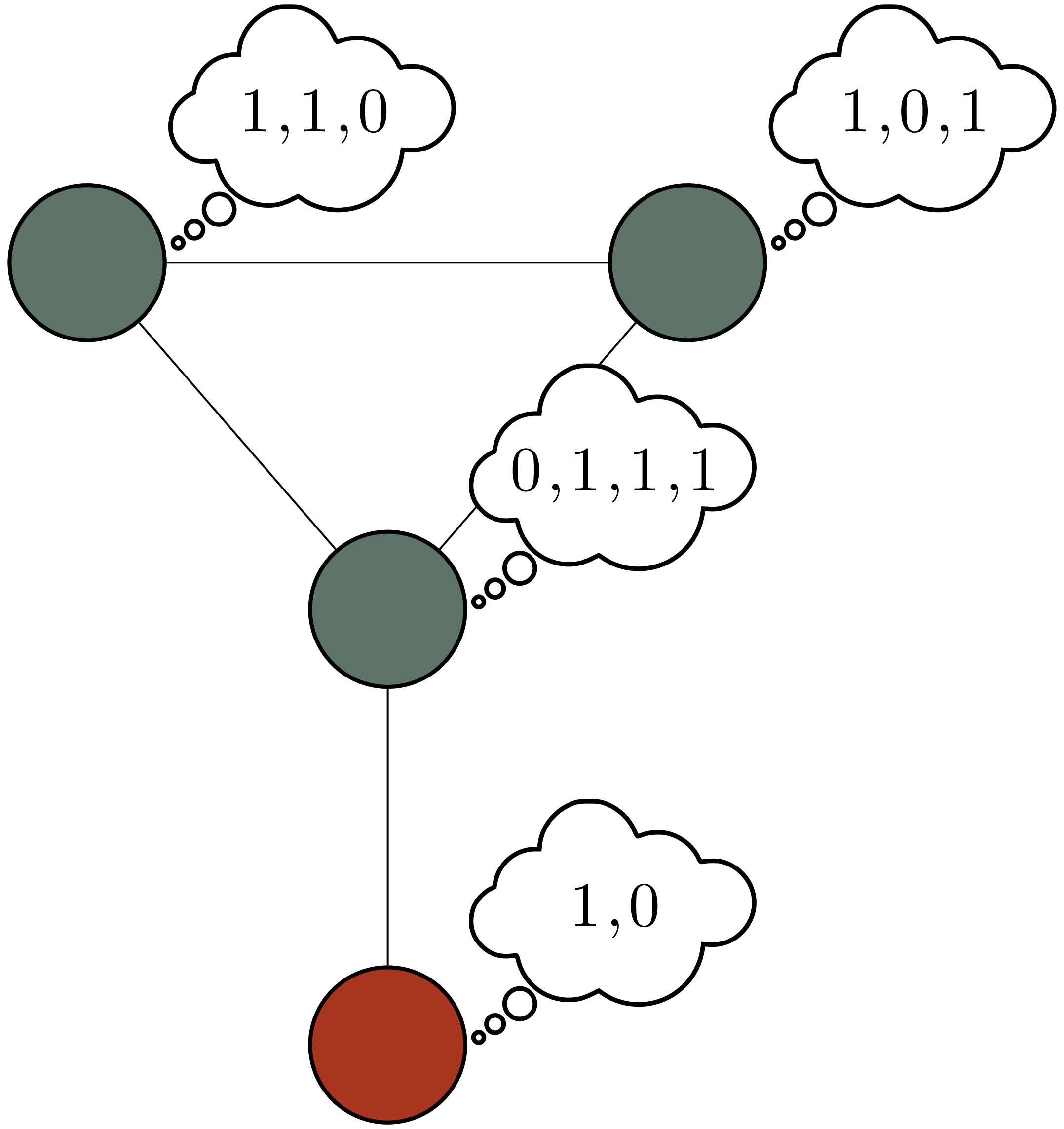

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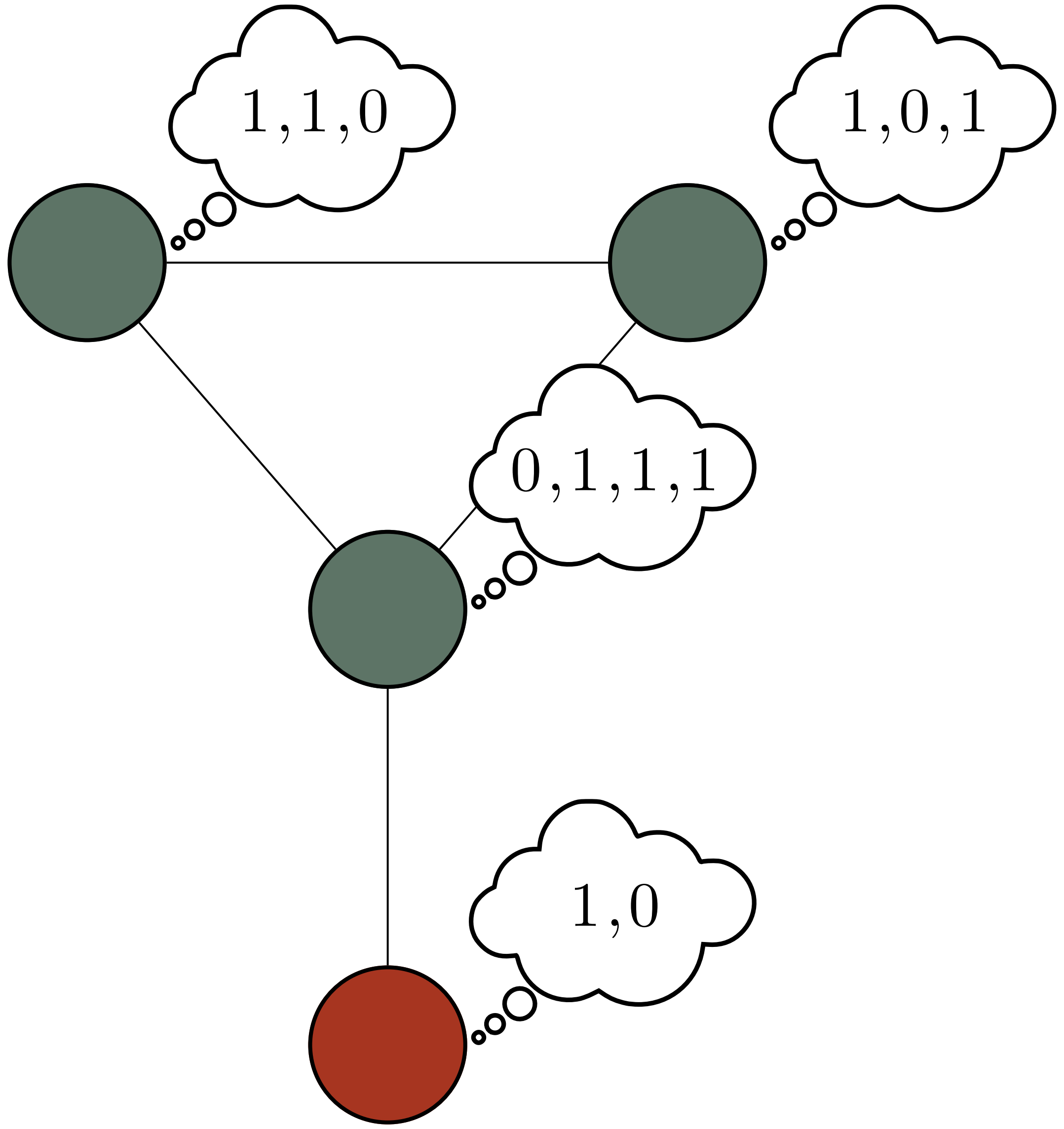
GIUSEPPE

Like jurors in a court case, sharing their evidence and thoughts.



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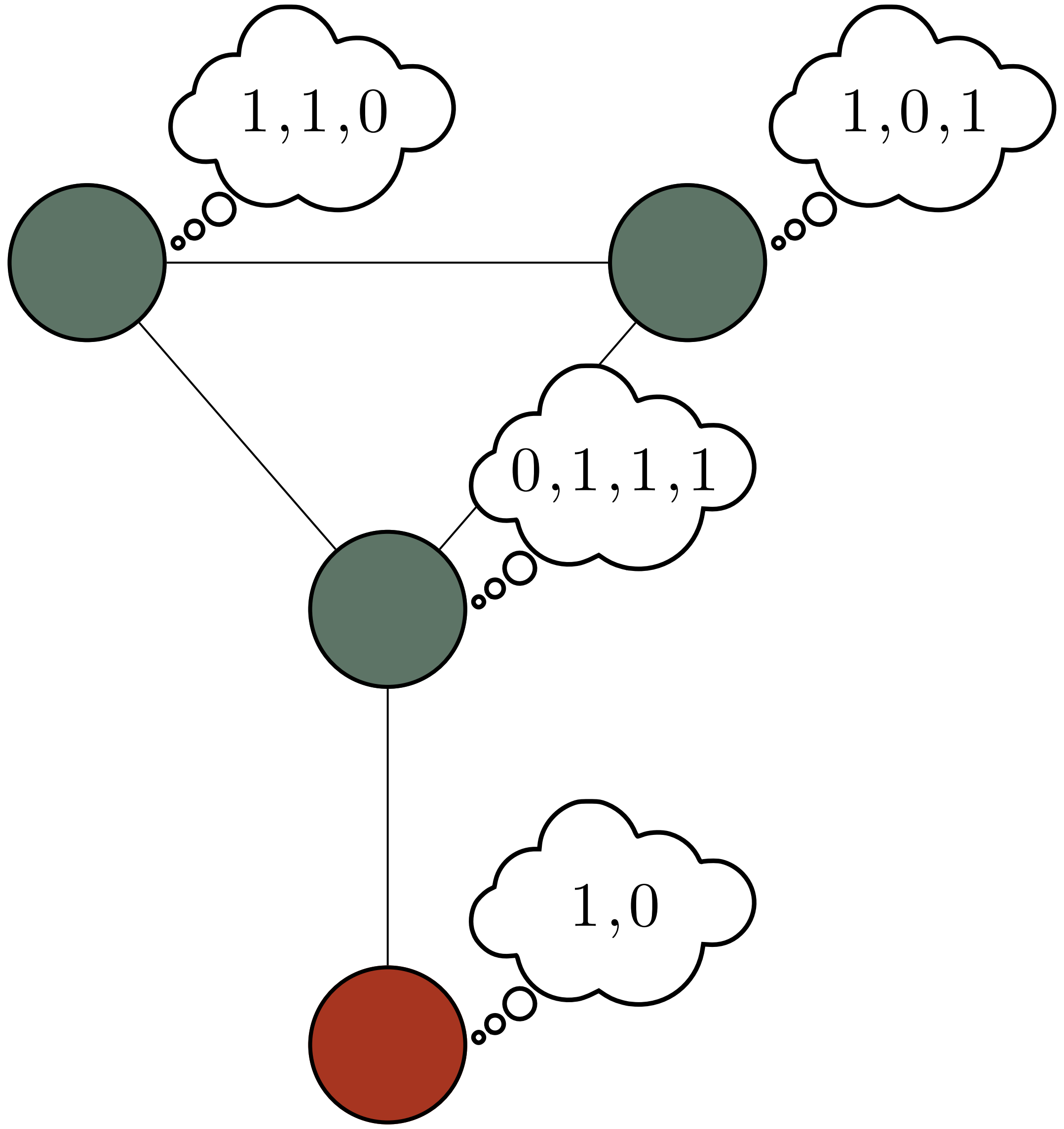

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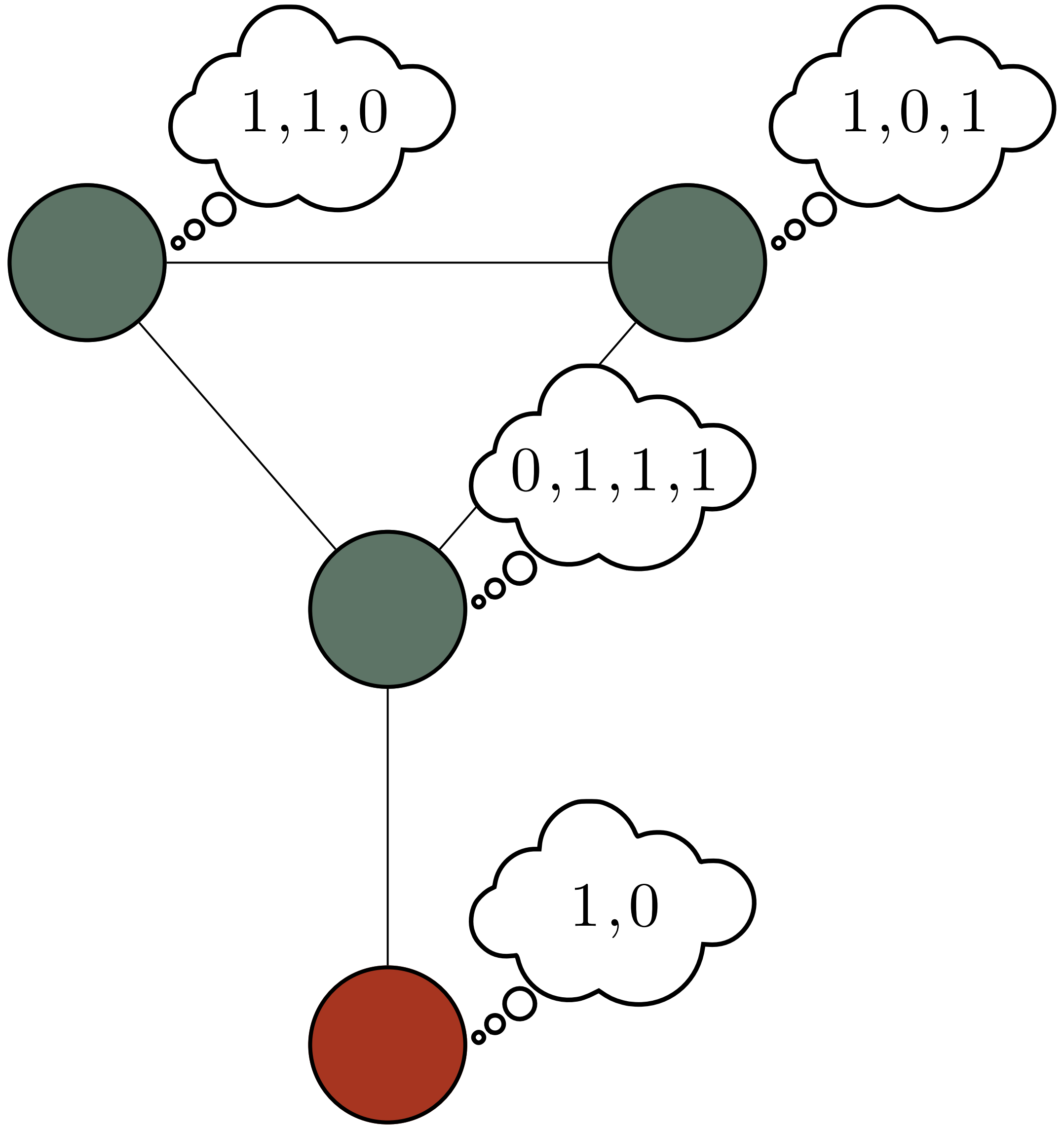
FREDERIK

But it results in correlated agents... and maybe more accurate decisions?



CONDORCET

Everyone gets more information, so everyone becomes more accurate, and the group gets even better...

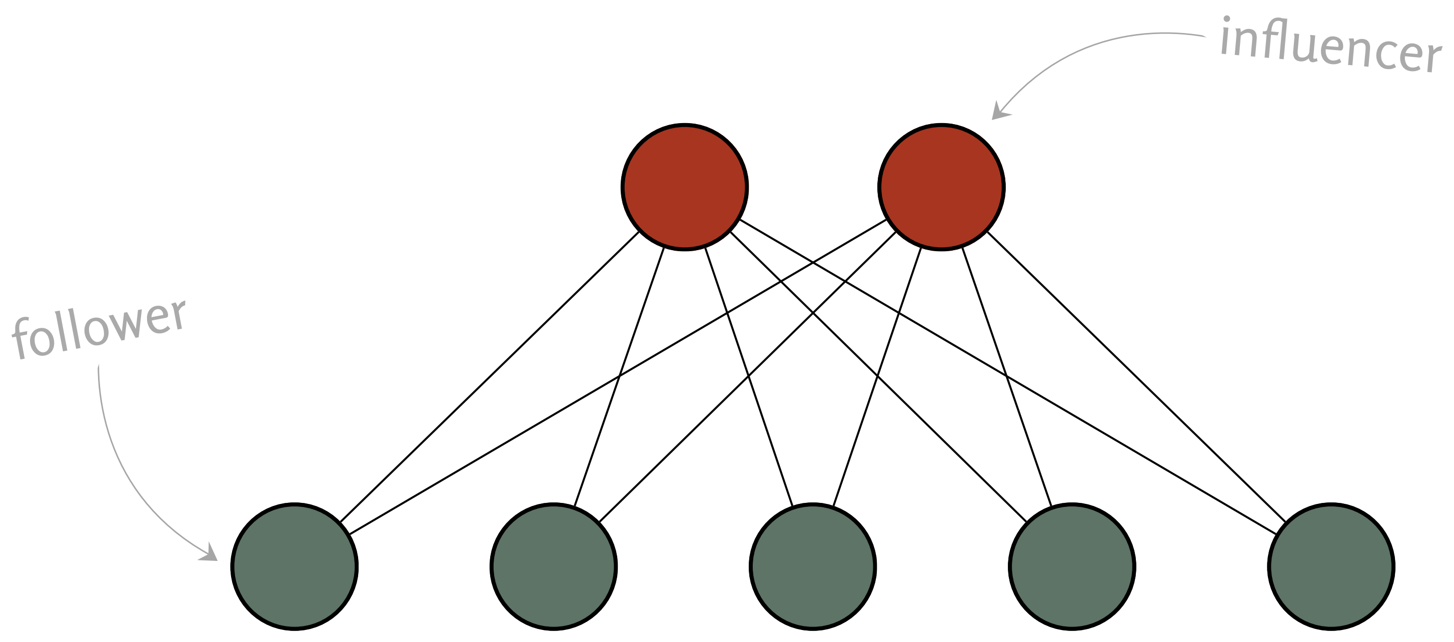
CONDORCET

Everyone gets more information, so everyone becomes more accurate, and the group gets even better…

… right?

Deliberation Gone Wrong

If the influencers get a wrong signal, followers end up believing the wrong thing.



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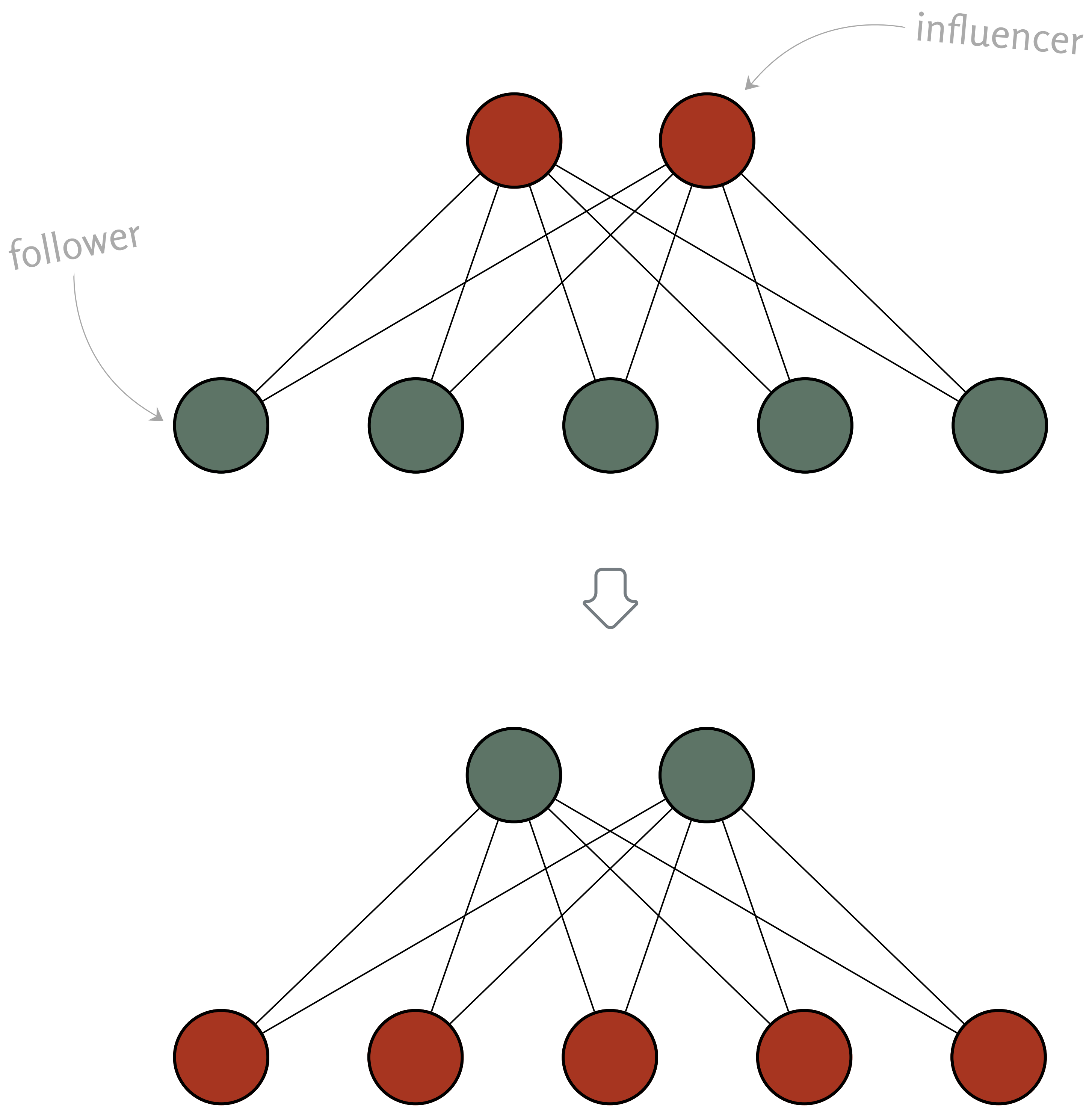

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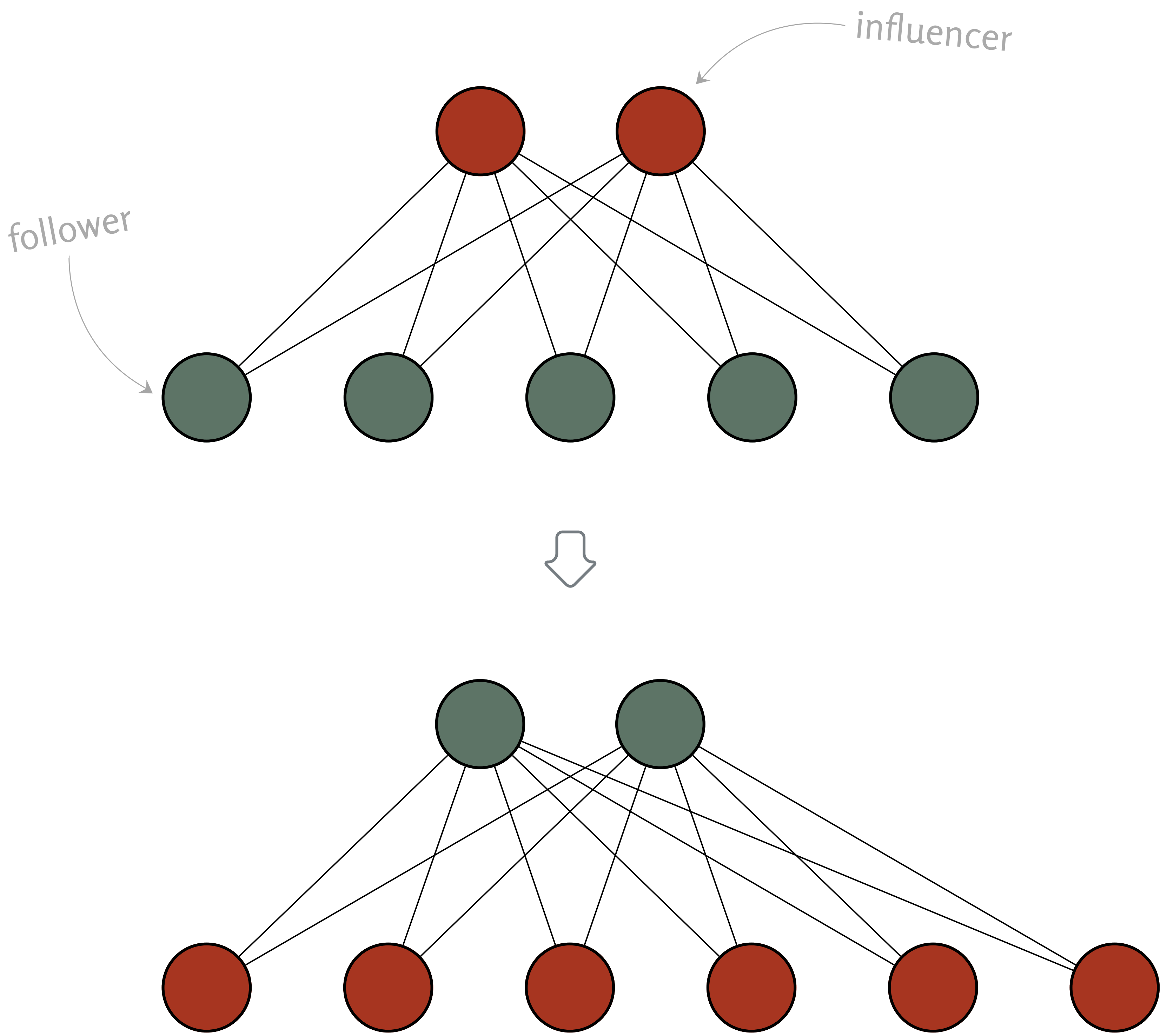

If the influencers get a wrong signal, followers end up believing the wrong thing.

The probability of a wrong group decision does *not* go down as we add more followers.*

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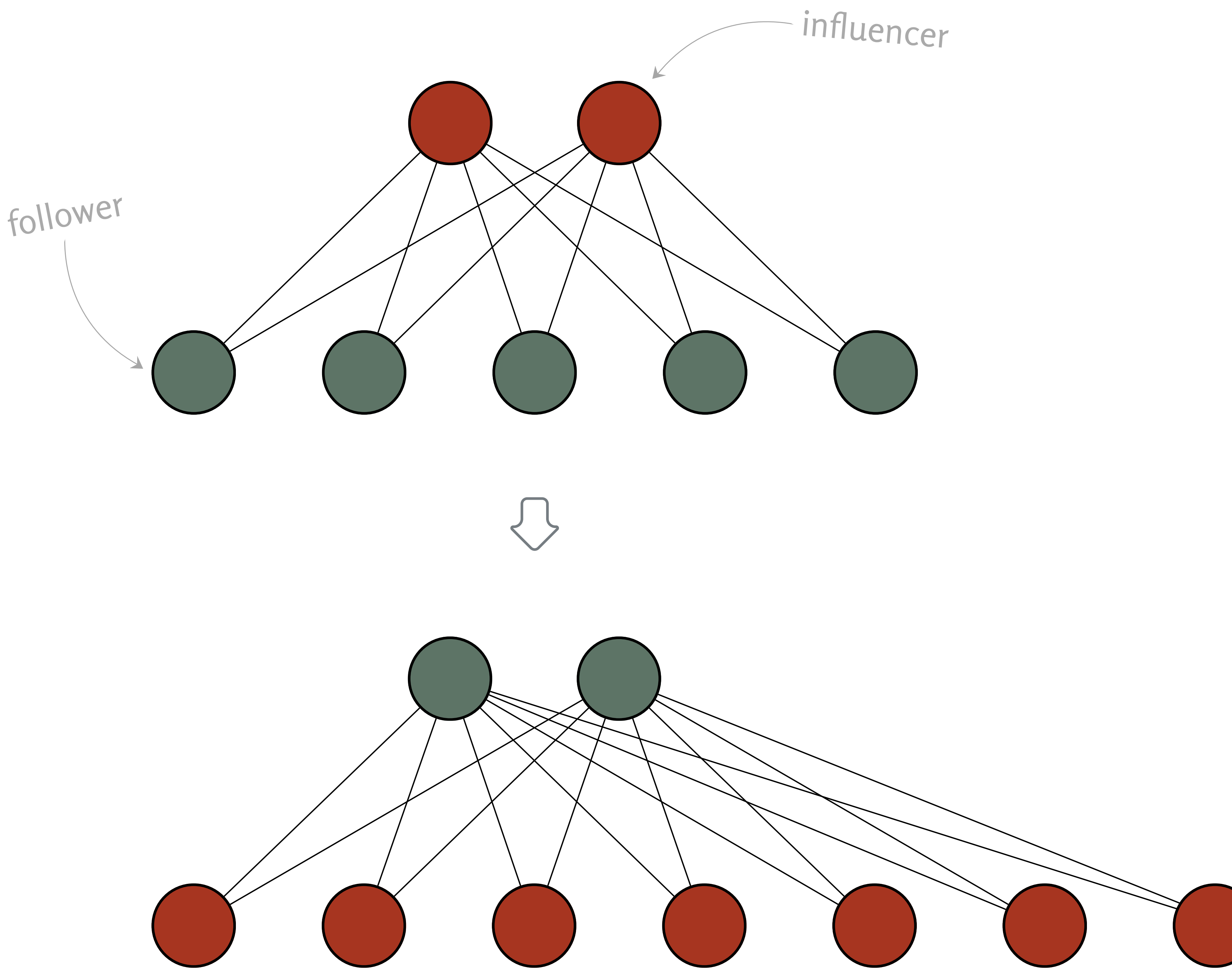

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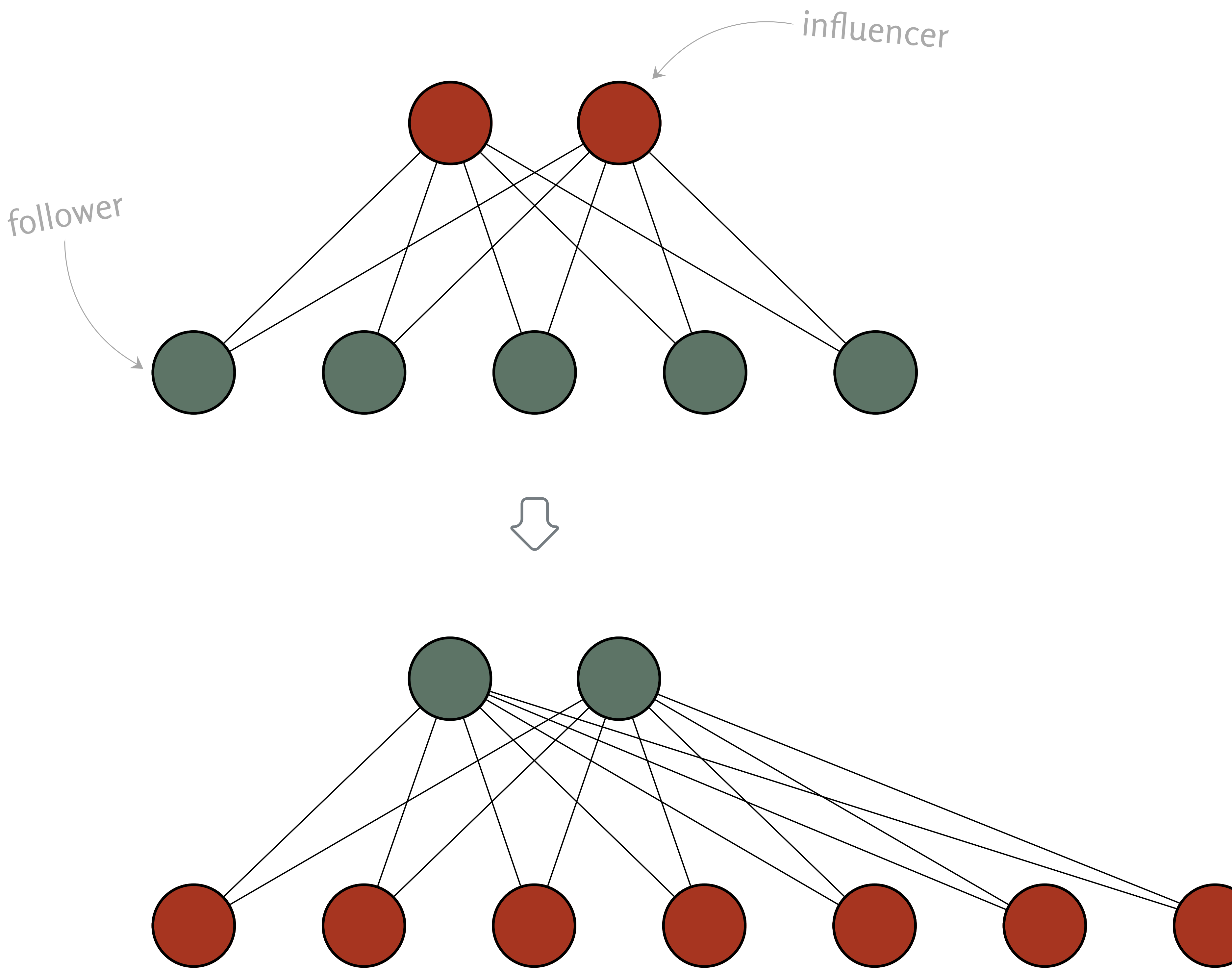
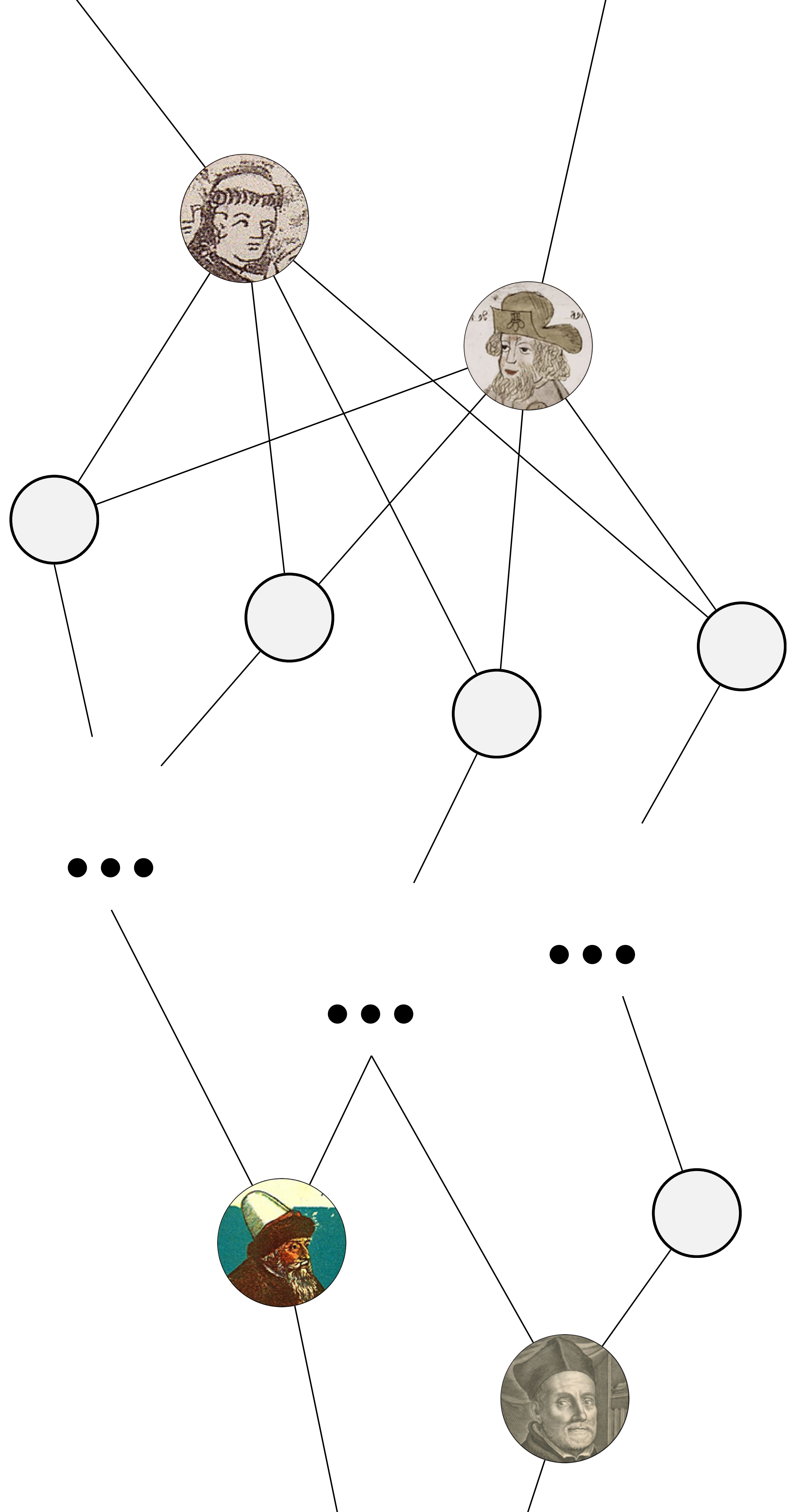


*This is an entire class of networks where deliberation keeps group accuracy below 1, even as the number of agents grows.

Presumably what happened with the vegetable lamb….



CONDORCET

Ok, but at least there are some cases in which deliberation helps...

CONDORCET

Ok, but at least there are some cases in which deliberation helps…

… right?

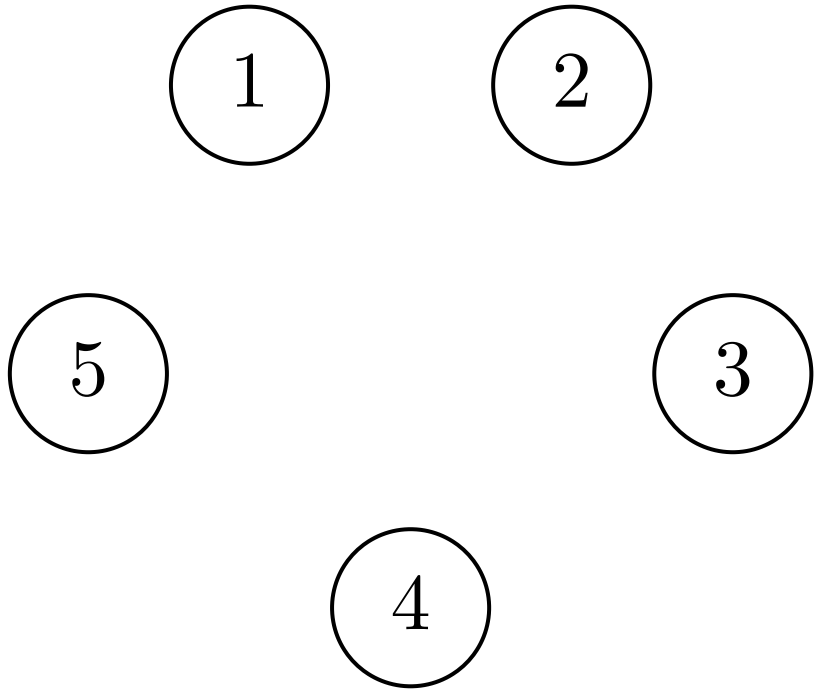
THEOREM

Group accuracy after deliberation via *any* graph G is never better than accuracy of direct voting.*

*direct voting = voting according to one's signal = deliberation via the empty graph

To see why, let's look first at the signal profiles that lead to correct decisions for direct voting, i.e., when there is no communication.

All signal profiles



11111

11110 11101 11011 10111 01111

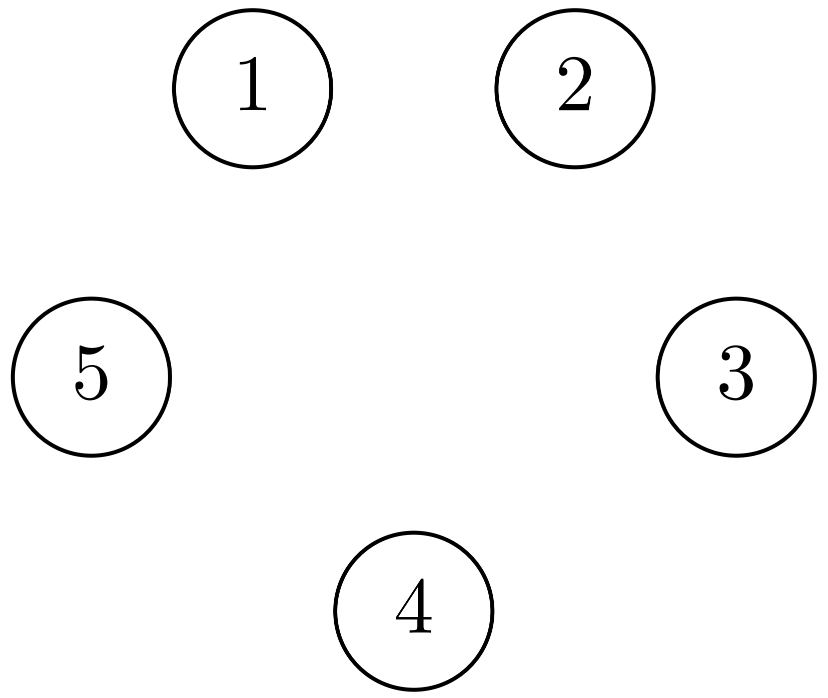
11100 11010 11001 10110 10101 10011 01011 01101 01110 00111

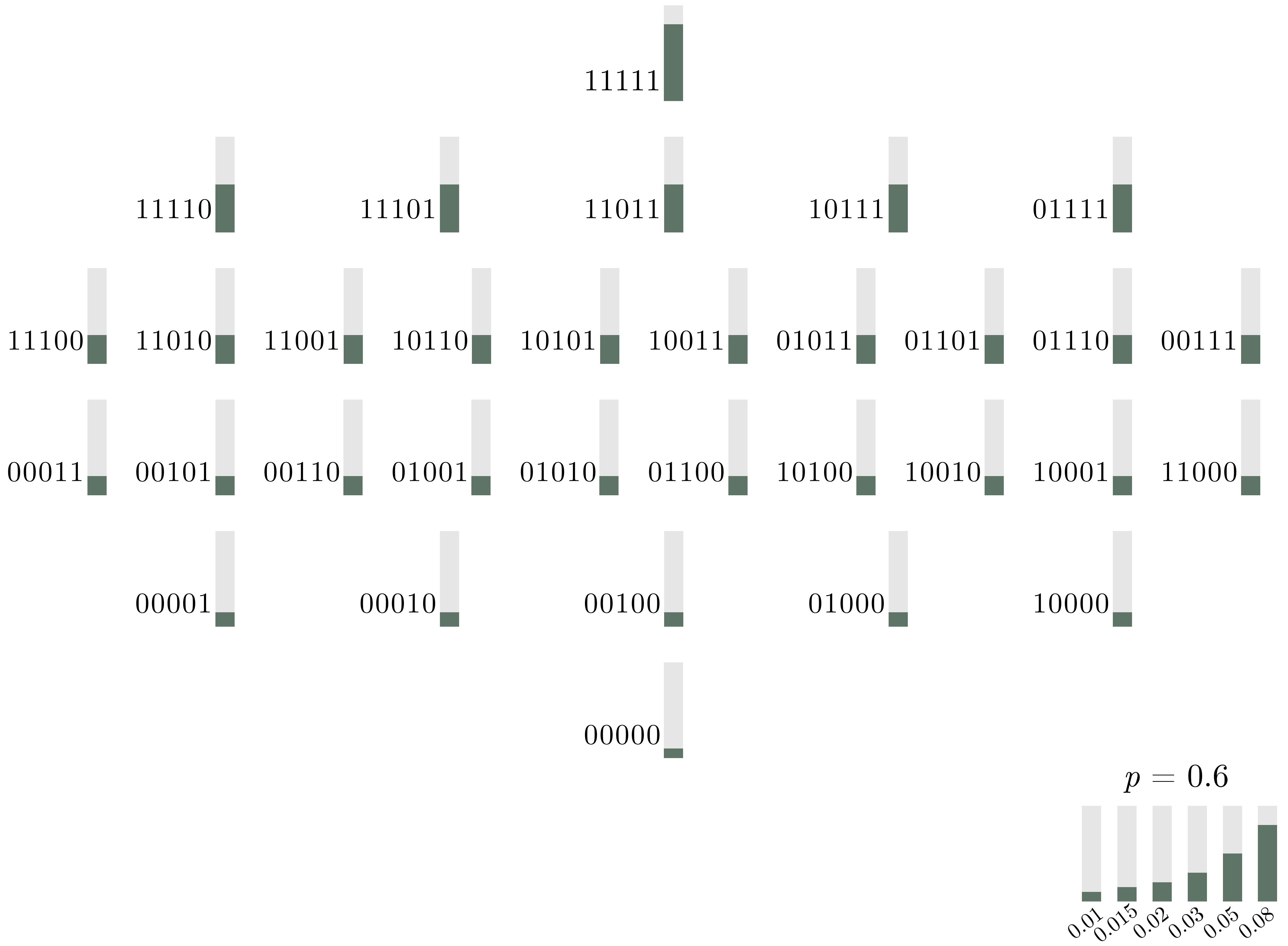
00011 00101 00110 01001 01010 01100 10100 10010 10001 11000

00001 00010 00100 01000 10000

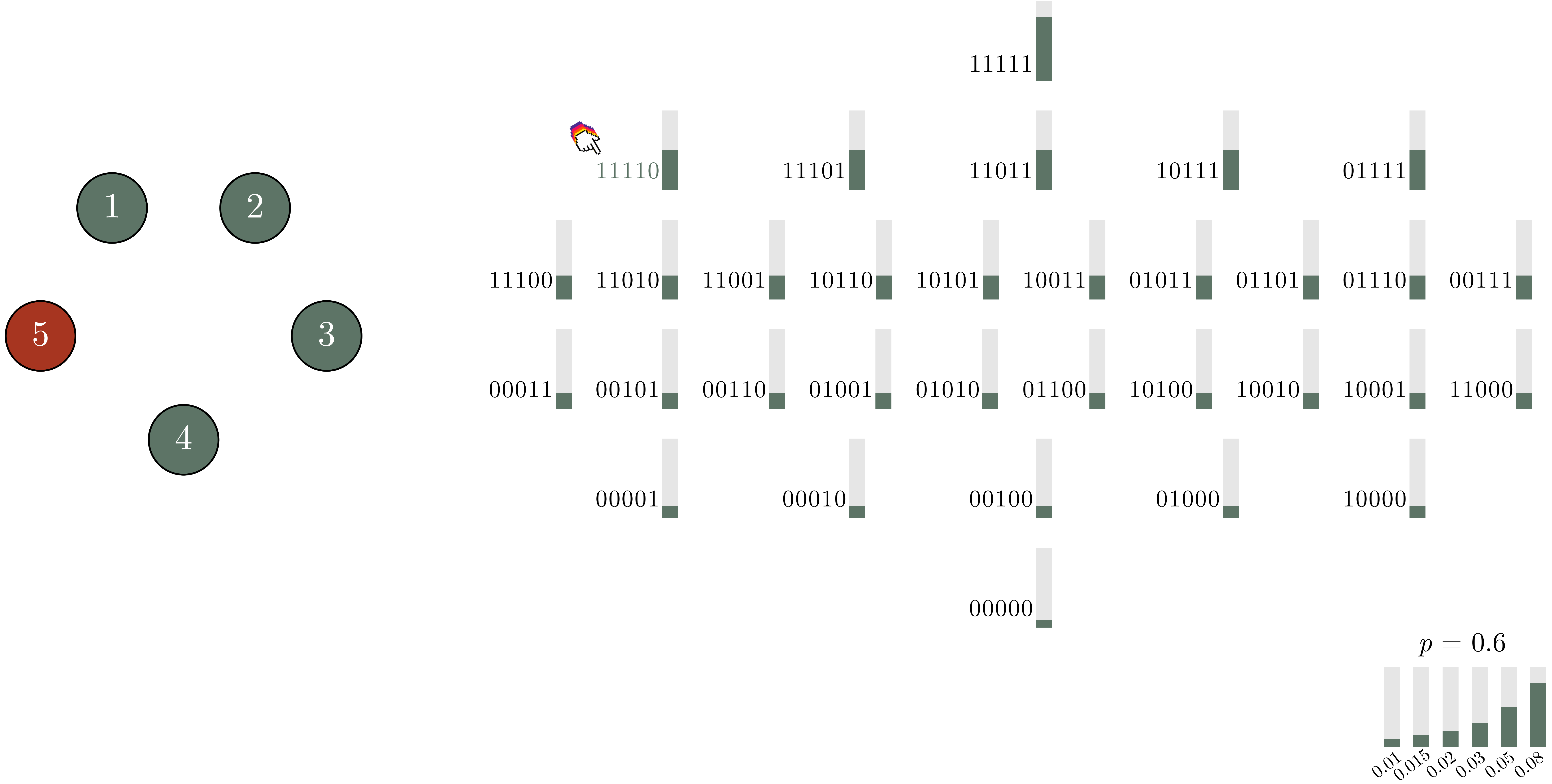
00000

All signal profiles





A signal profile that leads to a correct decision



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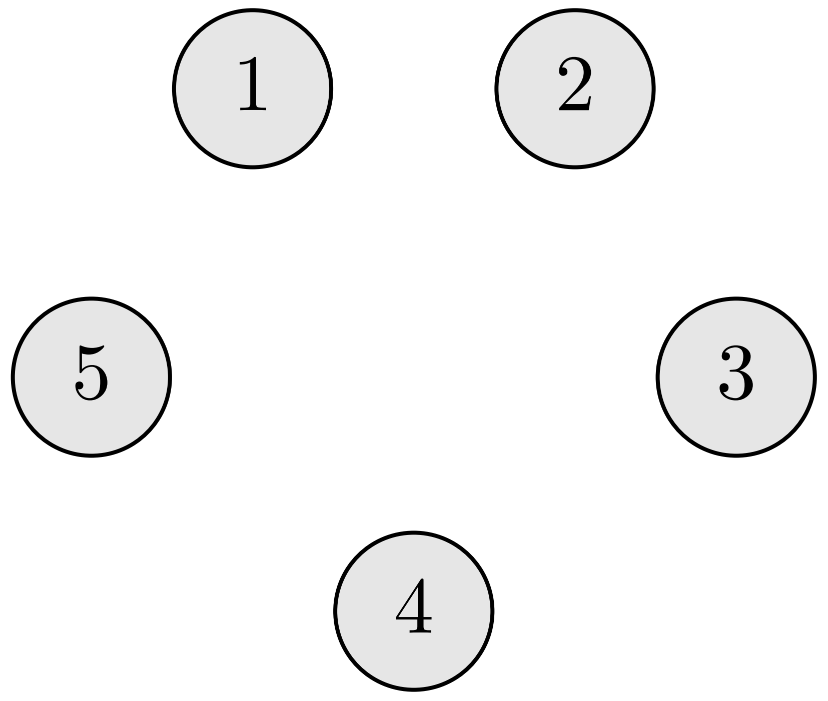
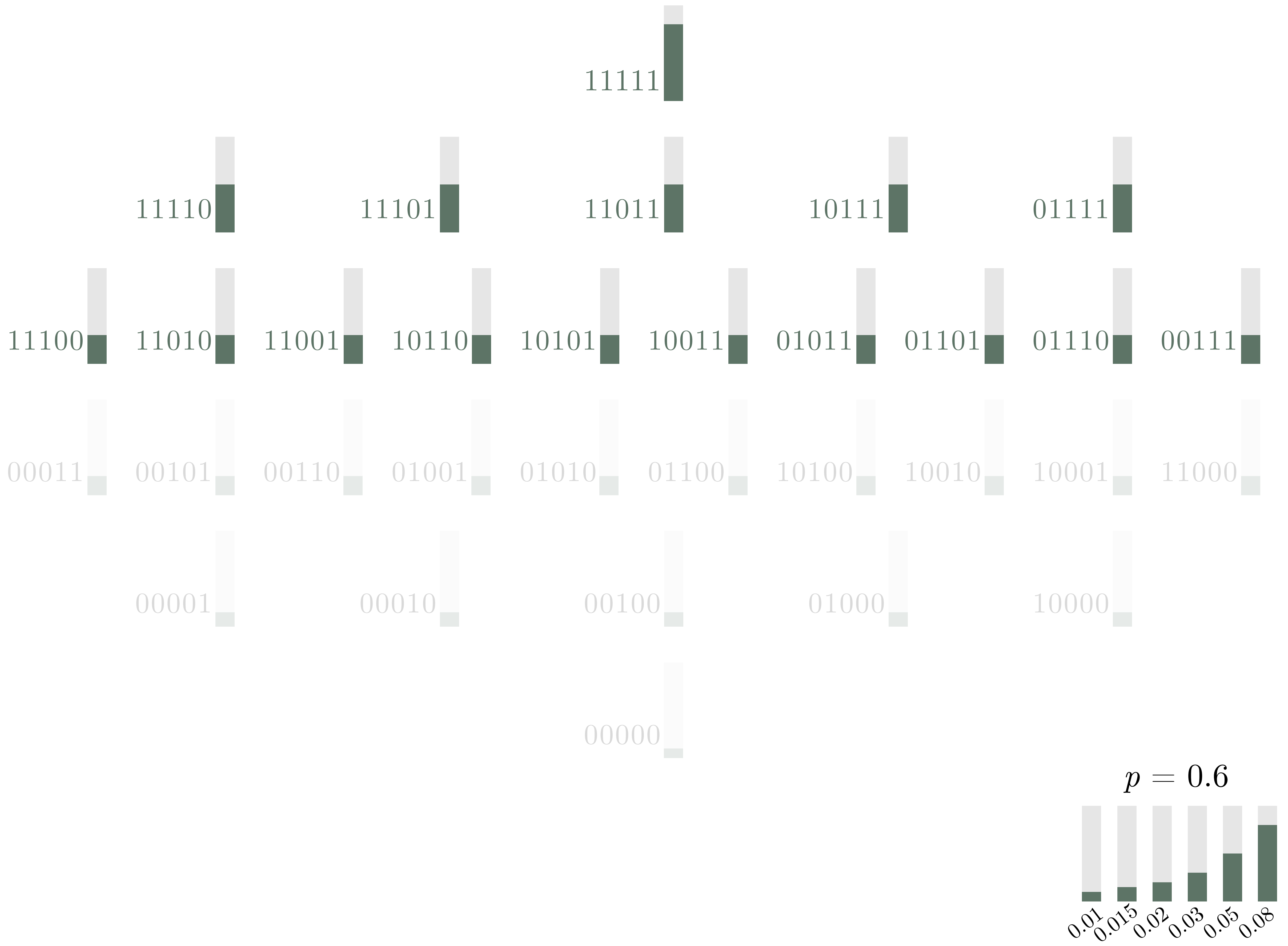
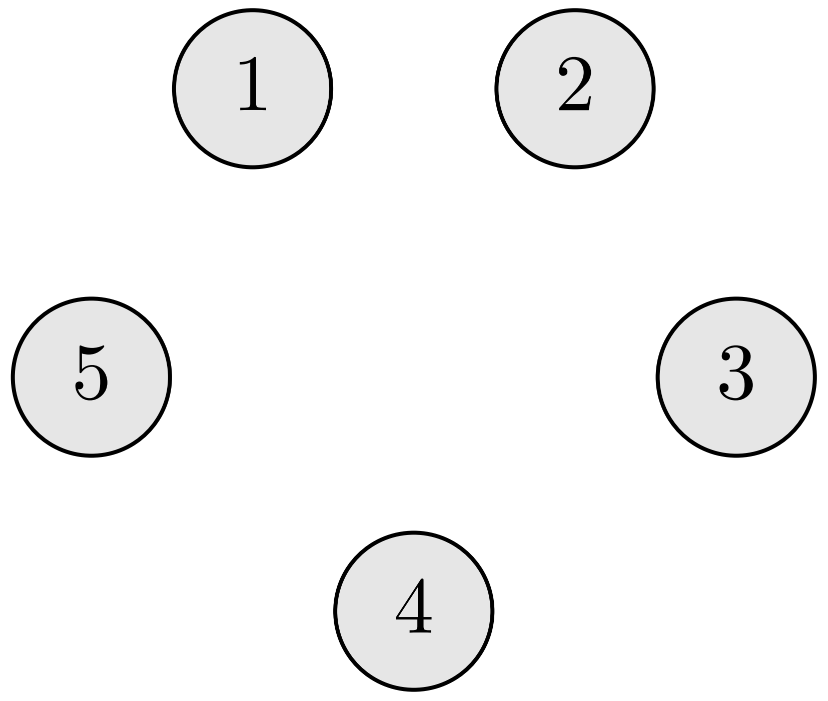
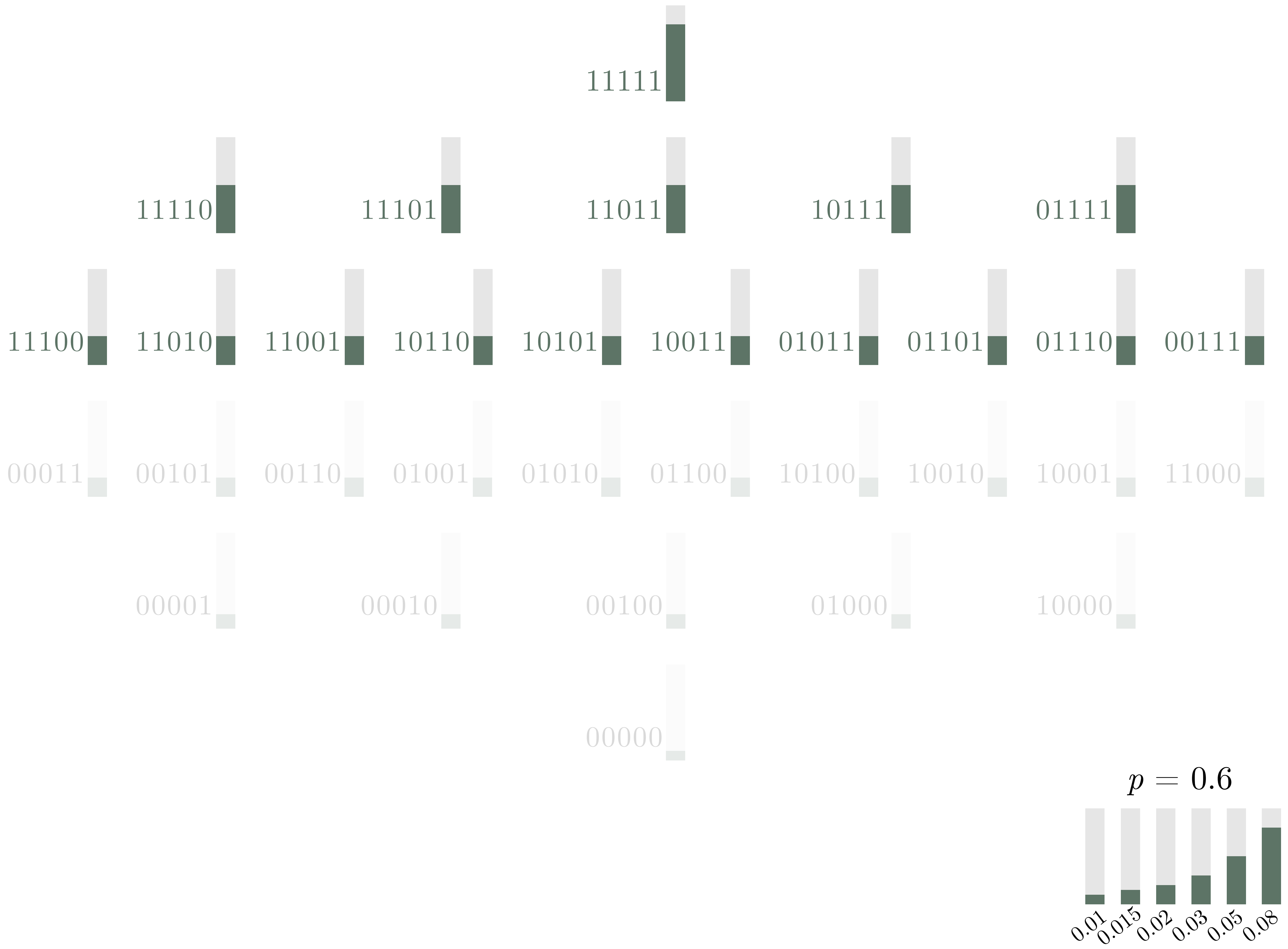




Let's compare this with the signal profiles that lead to a correct decision when some agents communicate with each other.

Let’s add some structure





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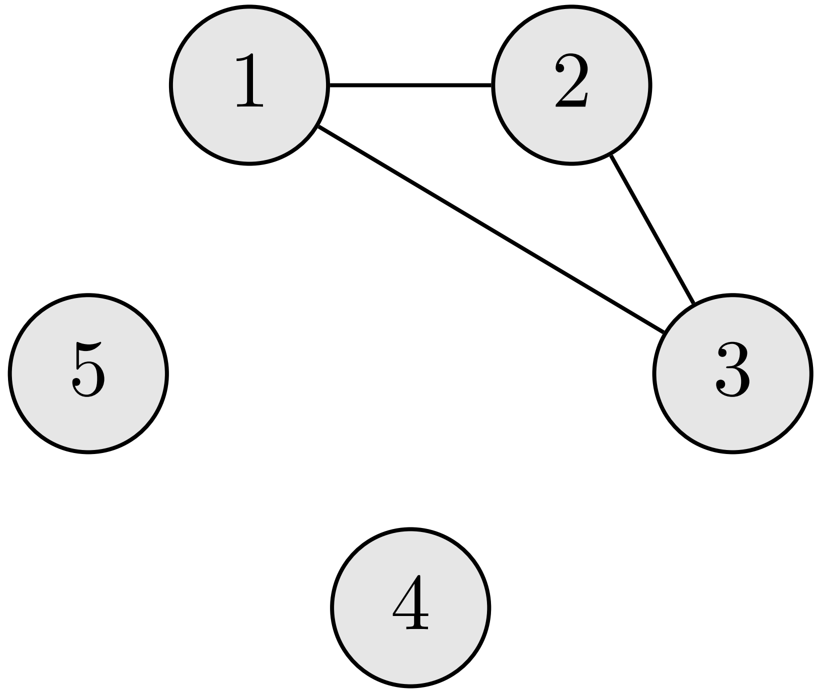
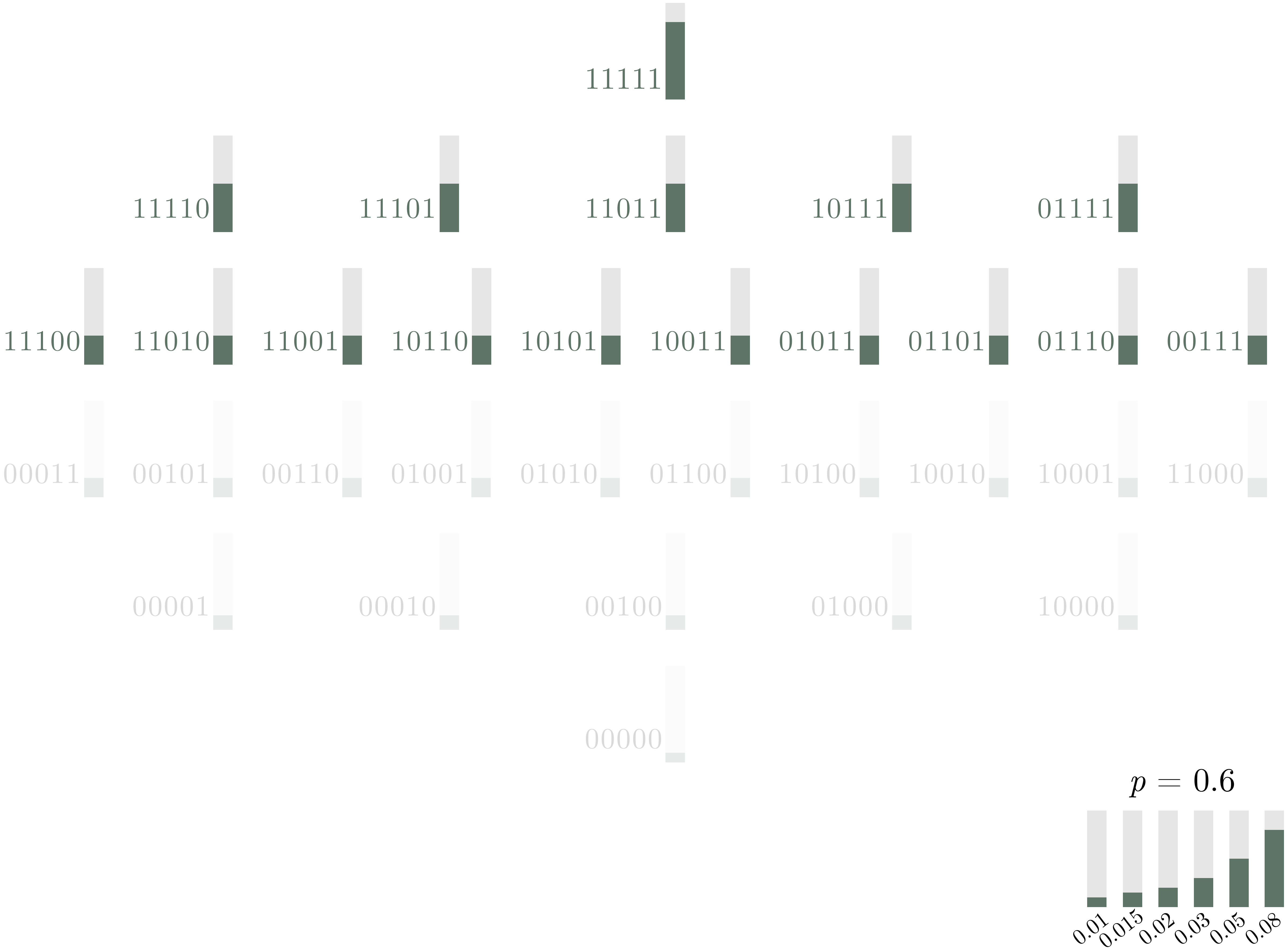
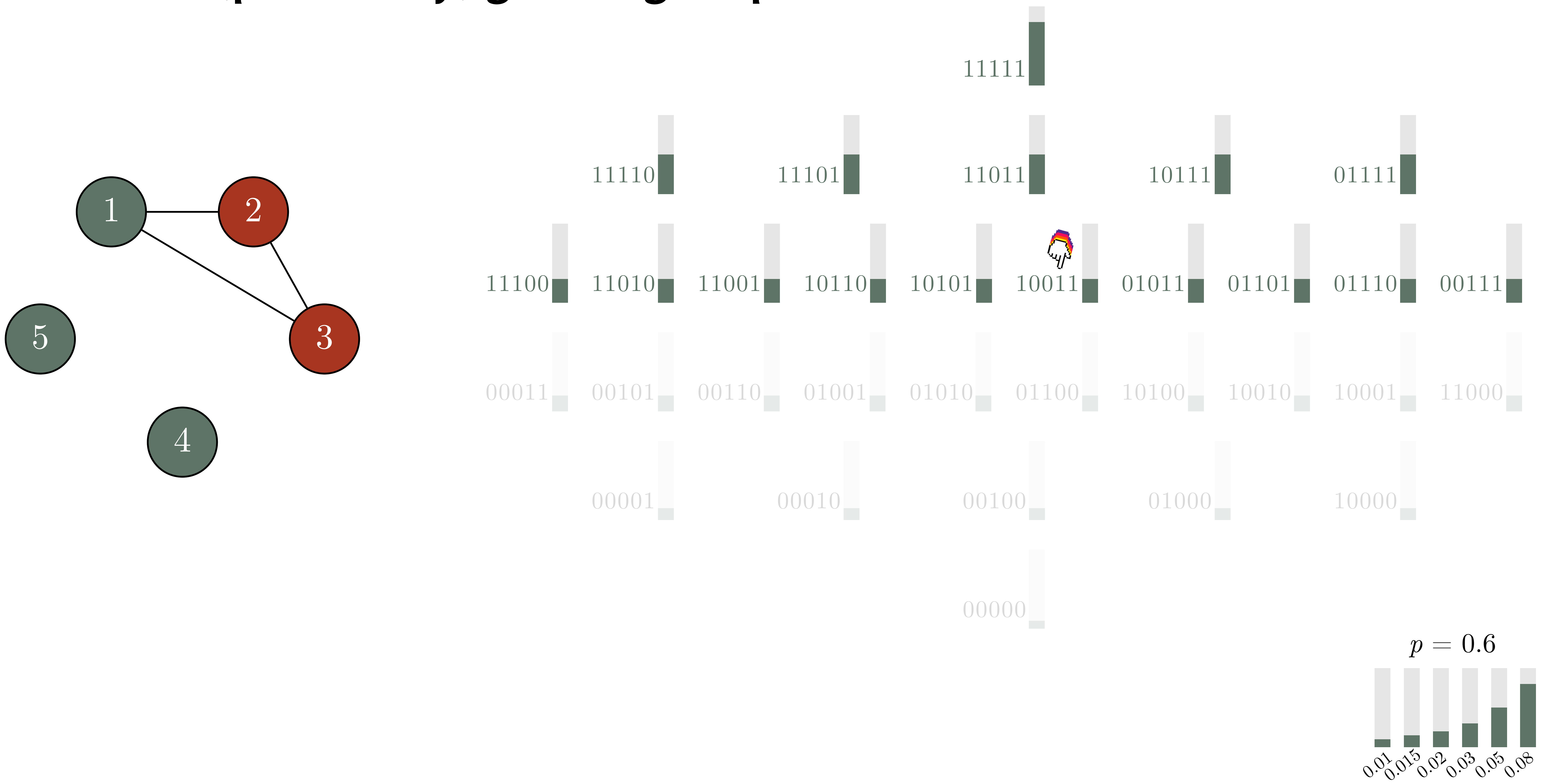
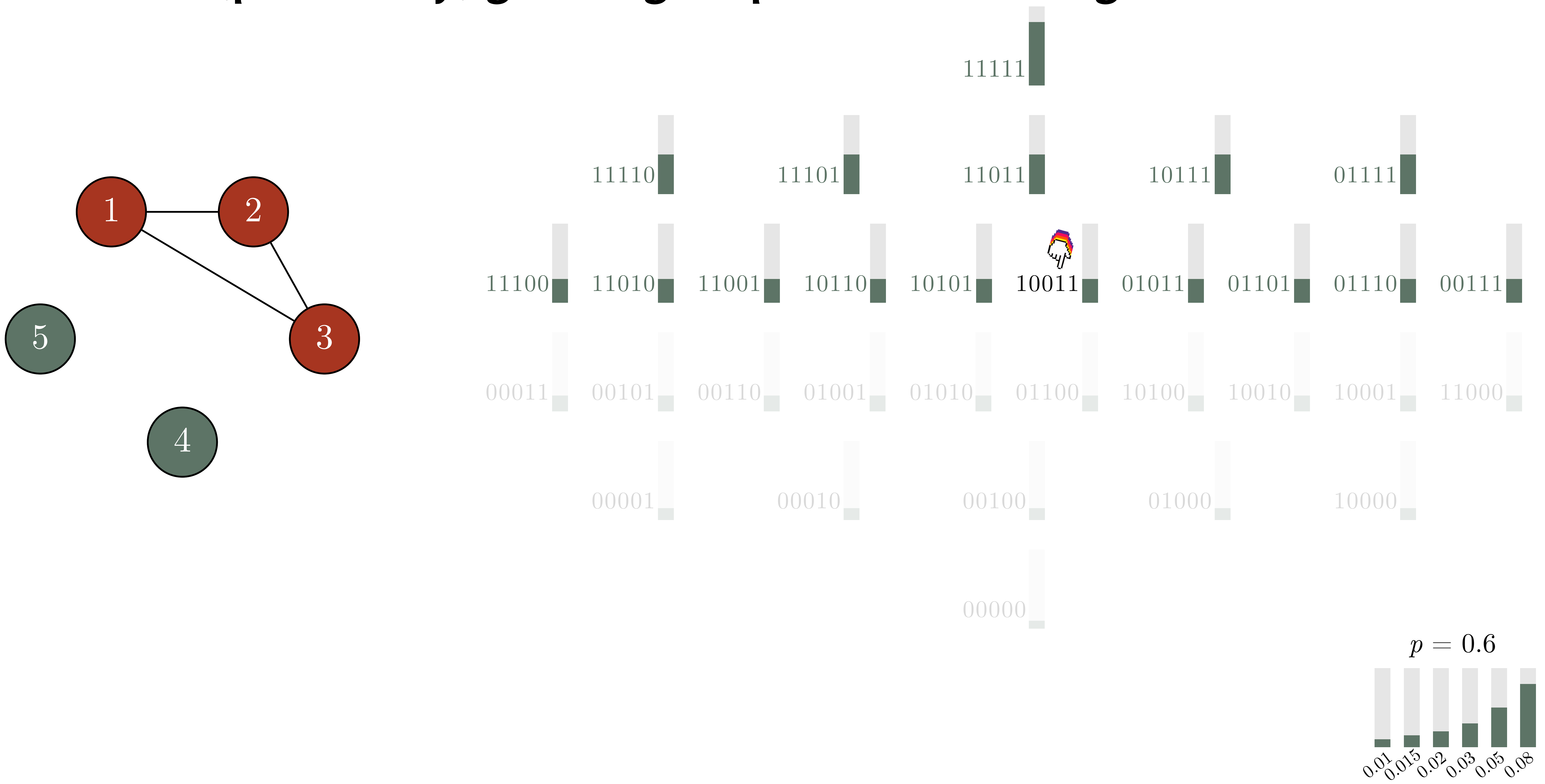




Take one (previously) good signal profile



Take one (previously) good signal profile and let agents deliberate



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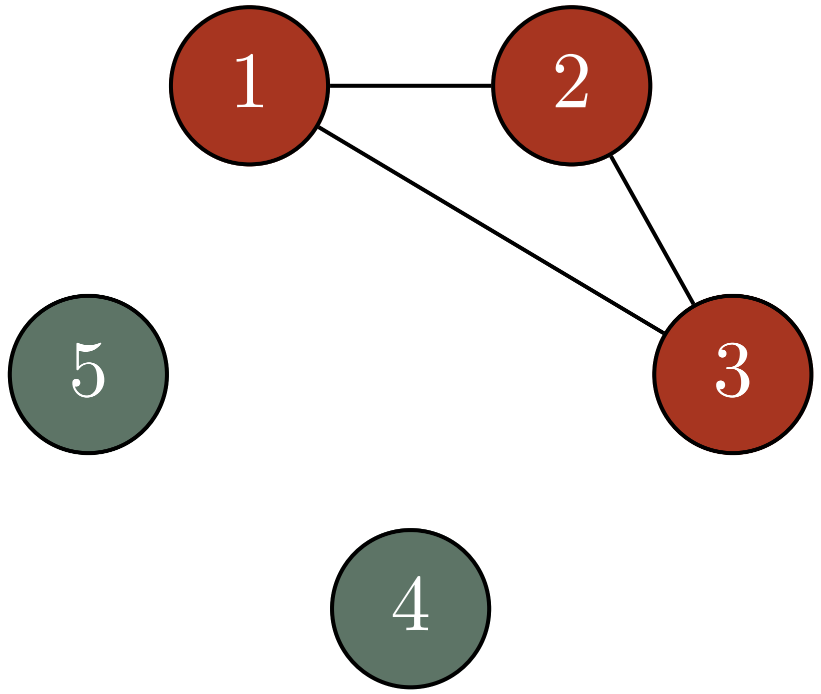
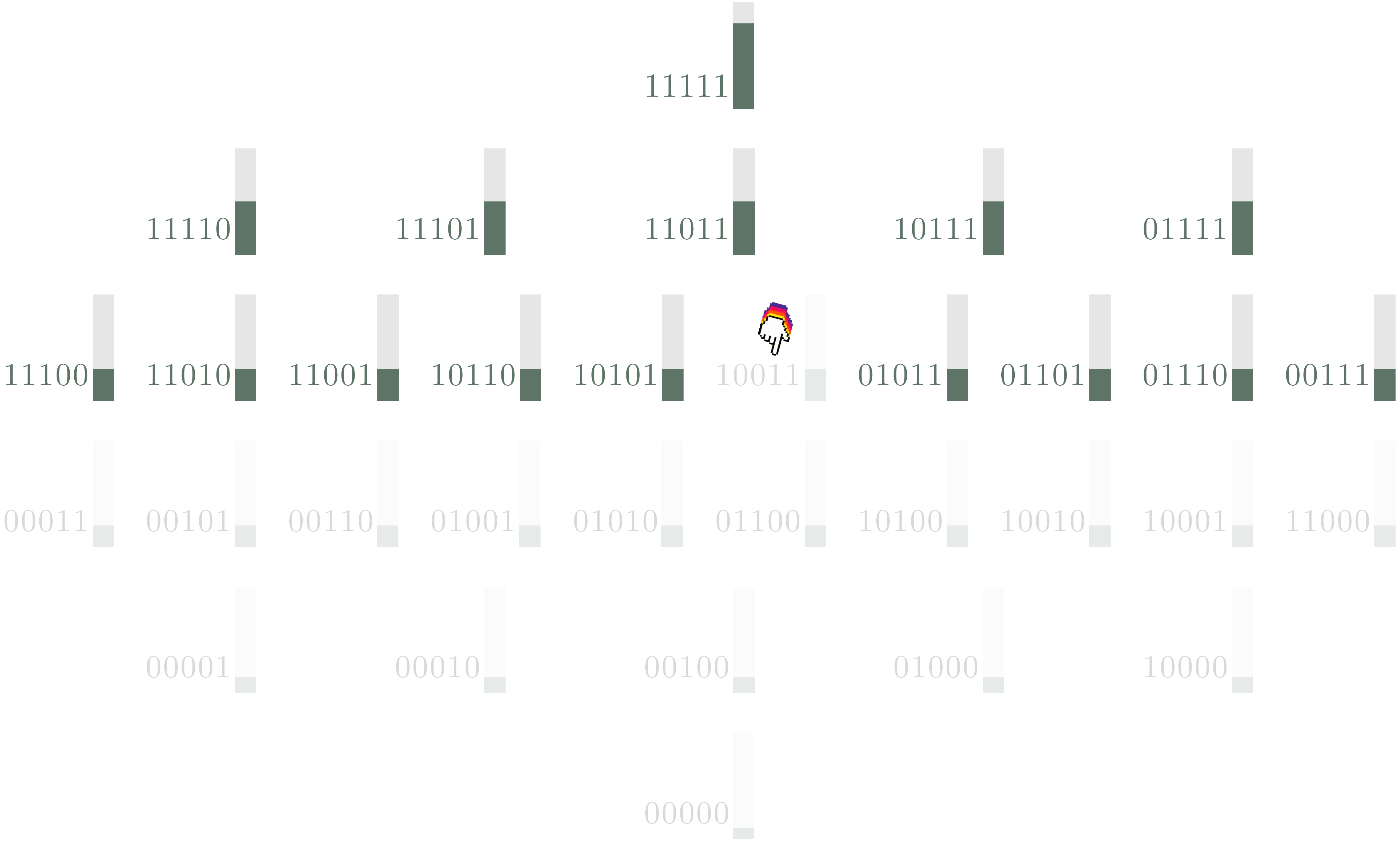
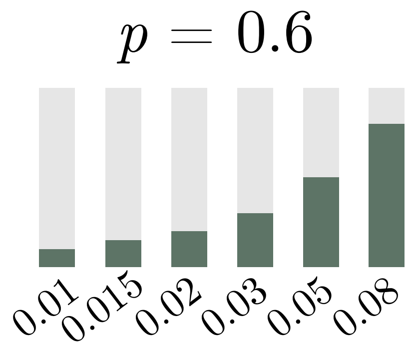
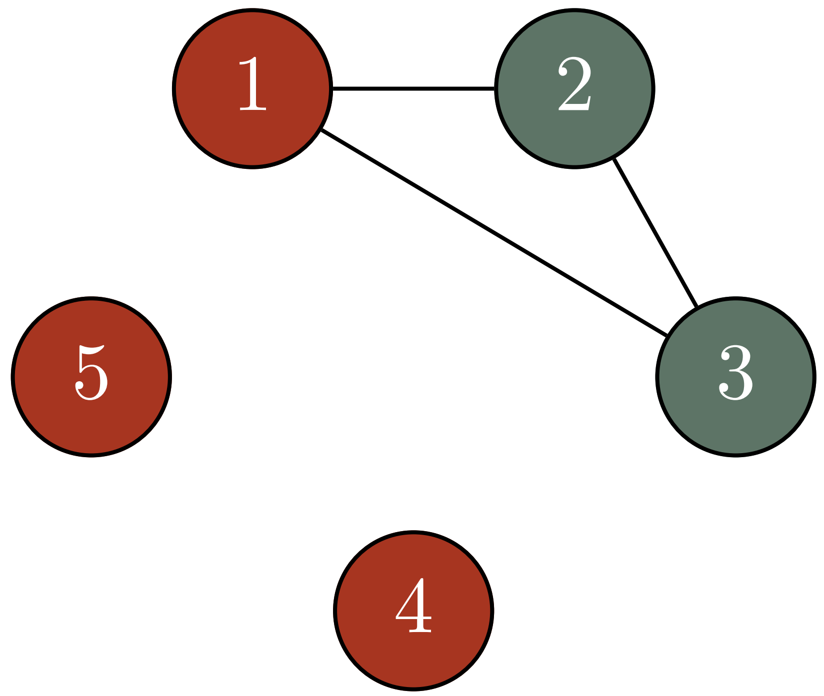


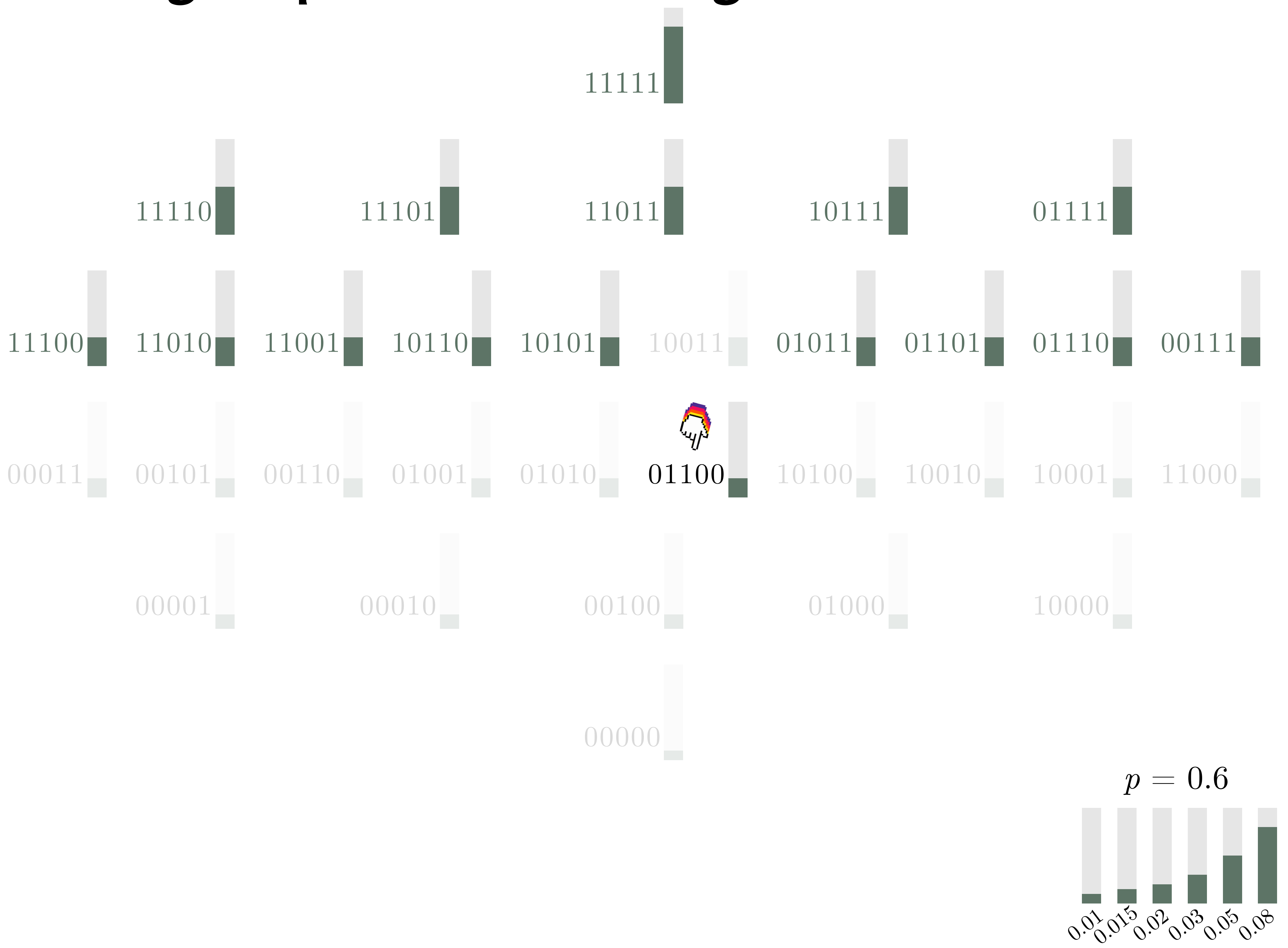
After deliberation the majority opinion is wrong!





Take one (previously) bad signal profile and let agents deliberate





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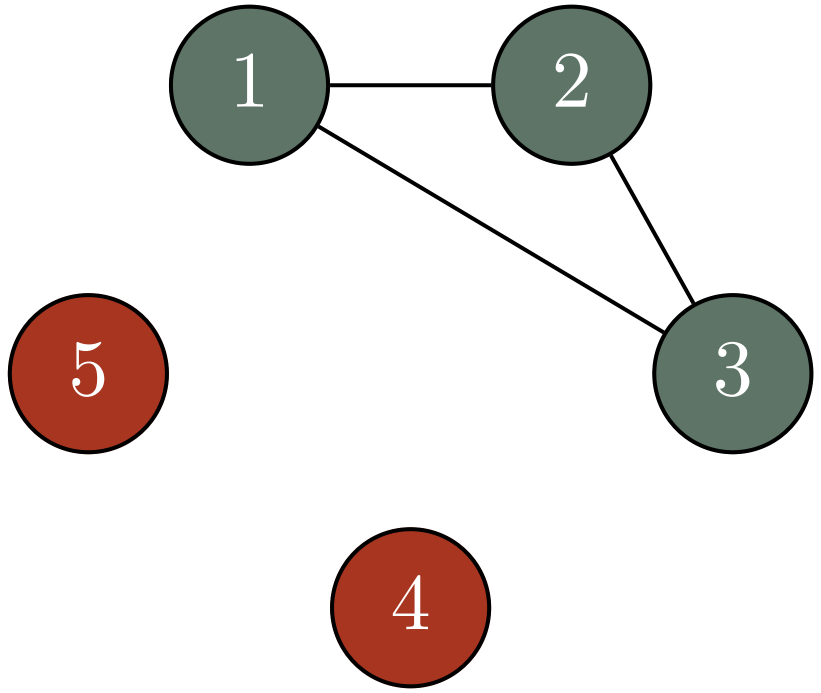
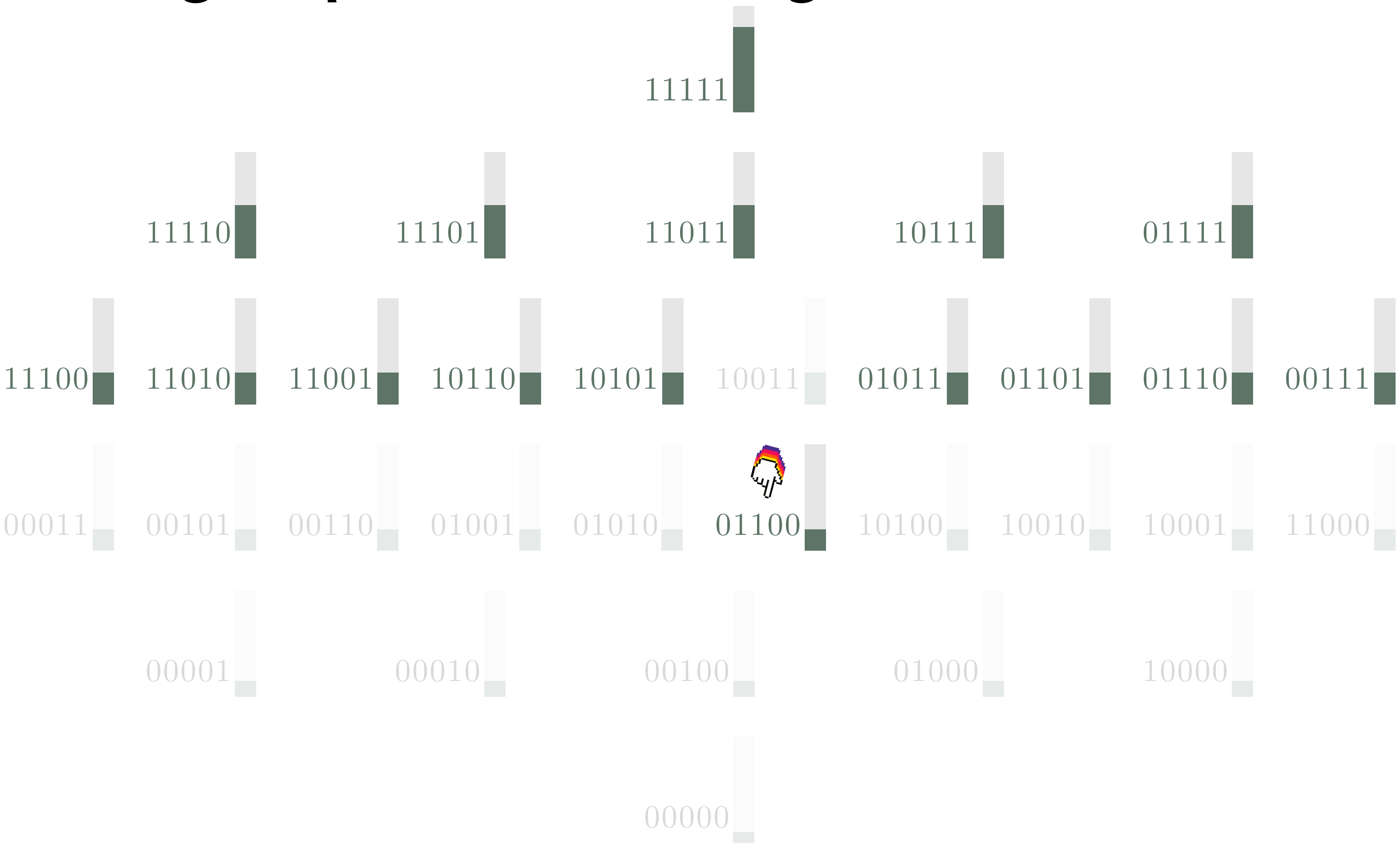
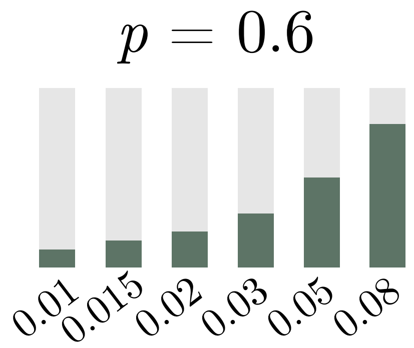




After deliberation the majority opinion is correct!



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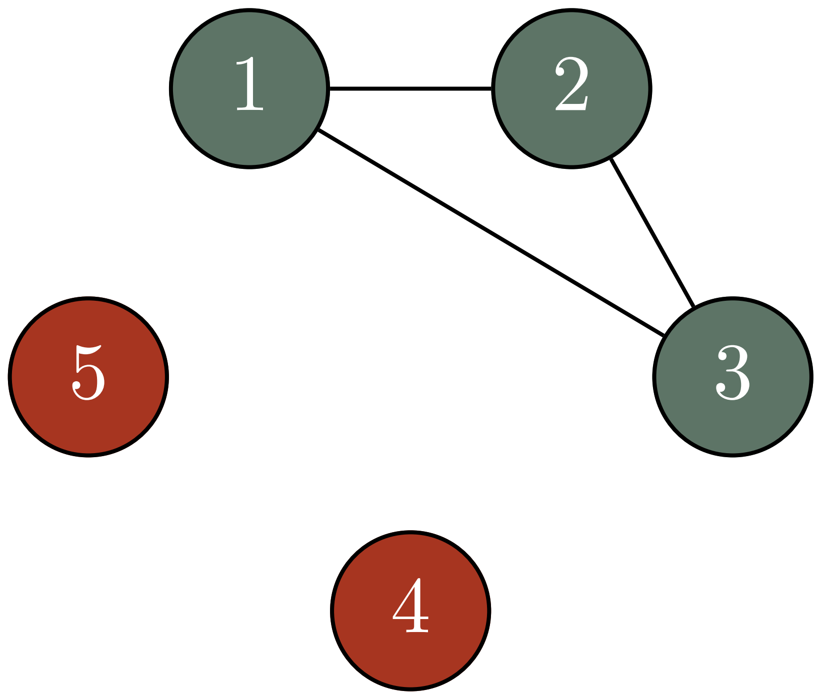
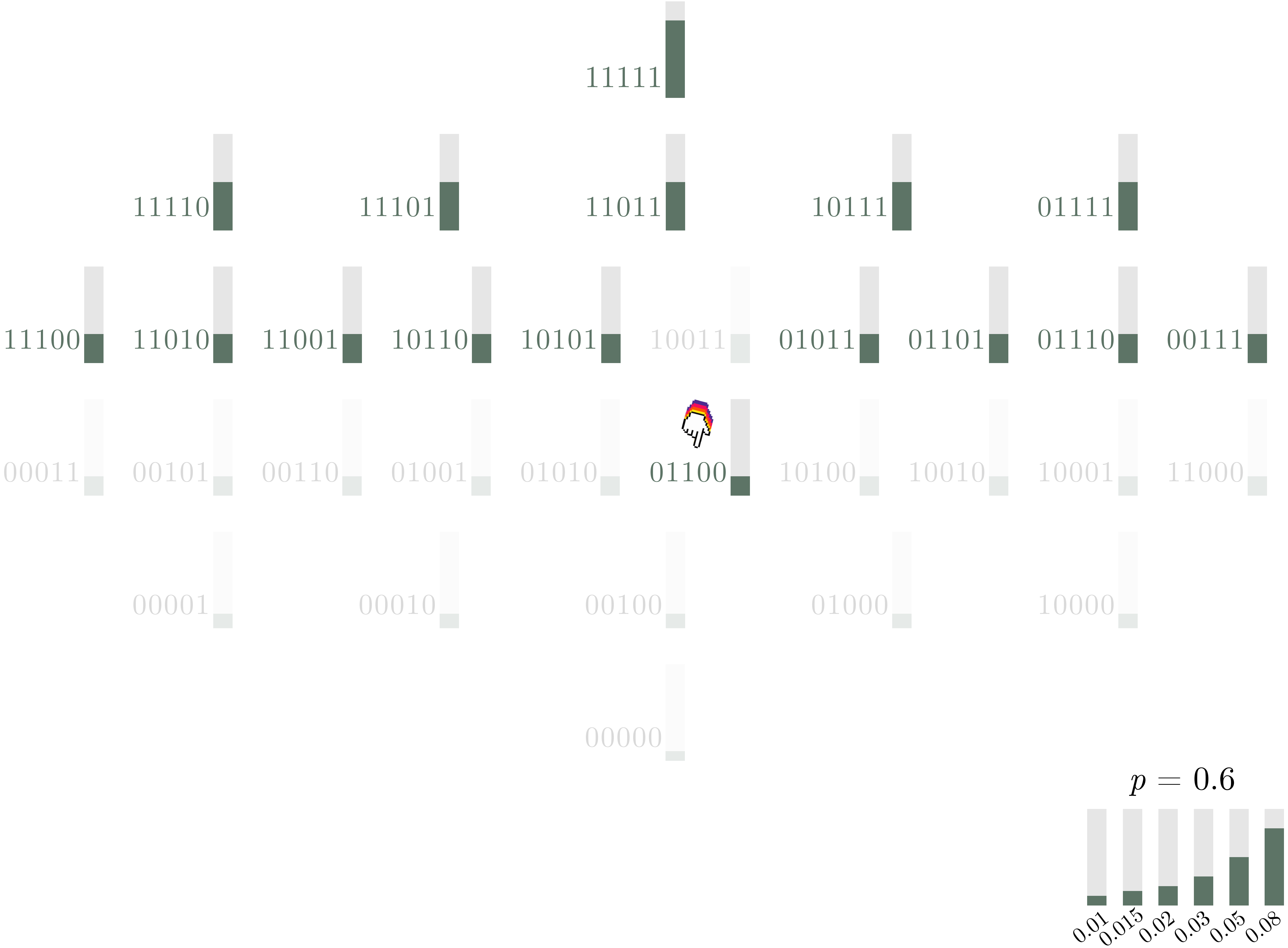


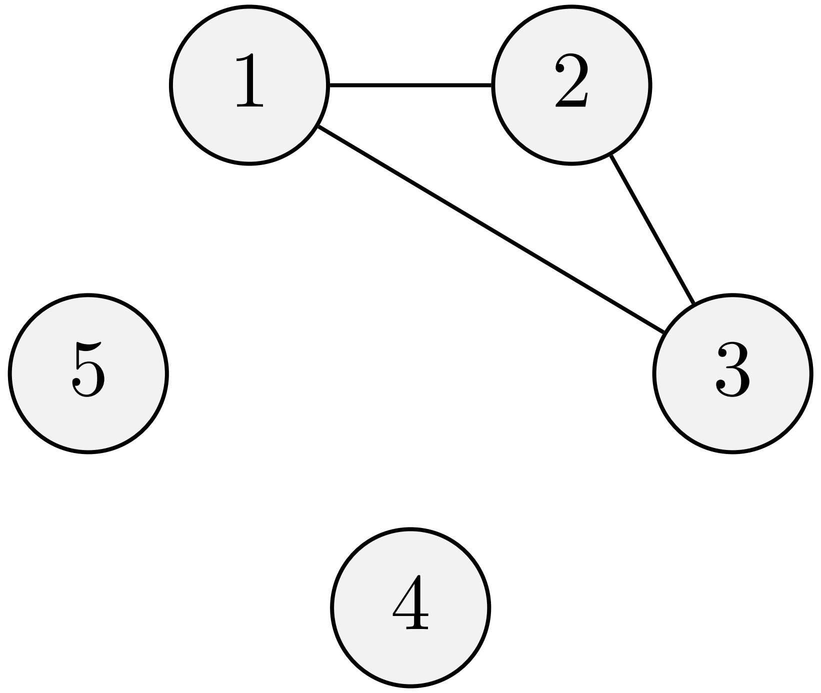


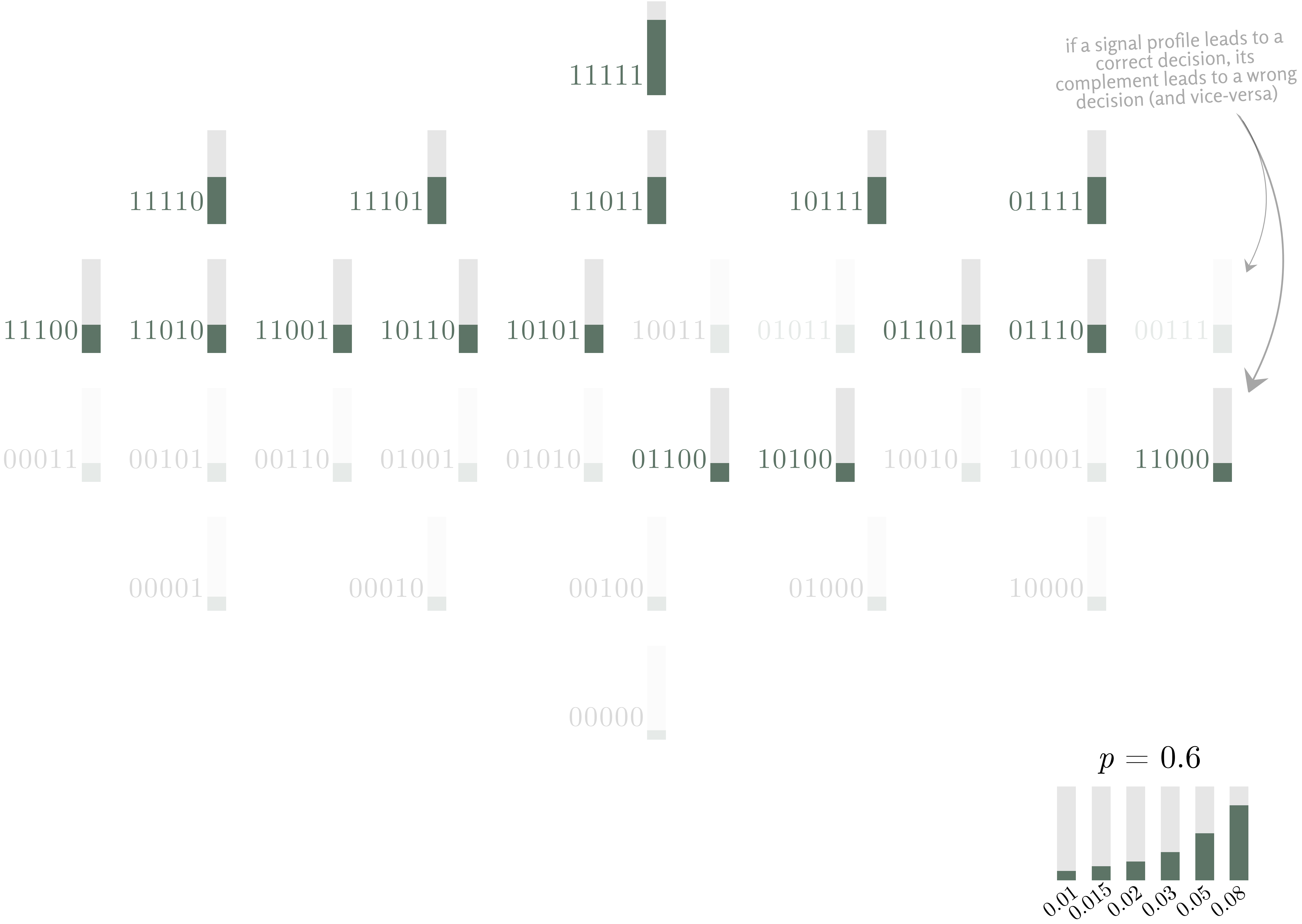
After deliberation the majority opinion is correct!

Interestingly, the two profiles we just looked at are complements of each other.

How often does this happen (on this graph)?

All good signal profiles on this graph





THEOREM

Group accuracy after deliberation via *any* graph G is never better than accuracy of direct voting.*

*direct voting = voting according to one's signal = deliberation via the empty graph

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PROOF

In general, adding structure to the graph you might end up trading a good (under the empty graph) signal profile for another one with slightly lower probability.

*direct voting = voting according to one's signal = deliberation via the empty graph

CONDORCET

Ok, but can we at least recover some asymptotic results?

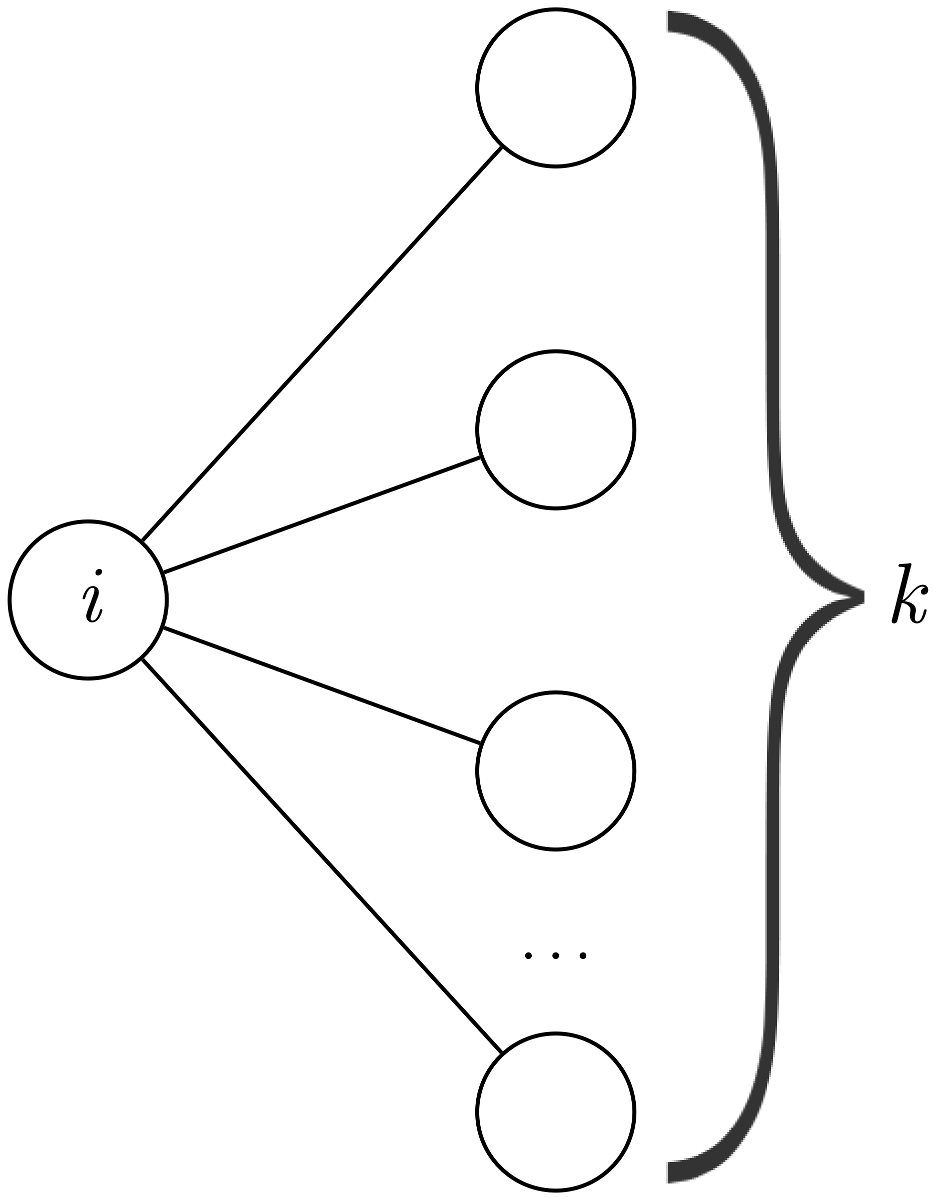
THEOREM

If G is a k-regular* graph on n nodes, with k even, group accuracy after deliberation via G approaches 1 in the limit, as n grows to infinity.

*every vertex has degree k

A wrong majority?

$$W_i = \begin{cases} 1, & \text{if } i \text{ is wrong after deliberation,} \\ 0, & \text{otherwise} \end{cases}$$



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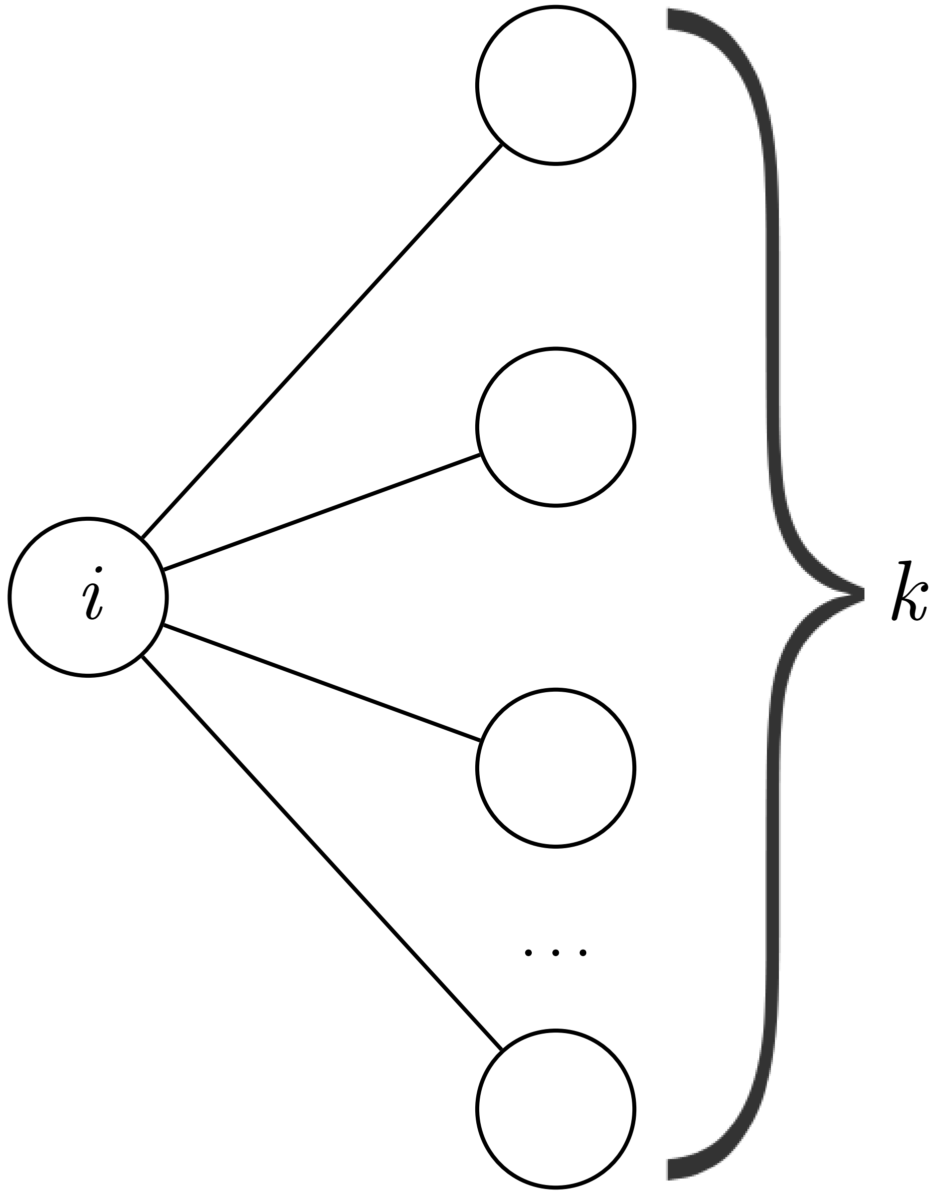
$$W_i = \begin{cases} 1, & \text{if } i \text{ is wrong after deliberation,} \\ 0, & \text{otherwise} \end{cases}$$

$$\Pr\left[W_i = 1\right] = m_k$$

probability that a majority of k+1 signals are wrong



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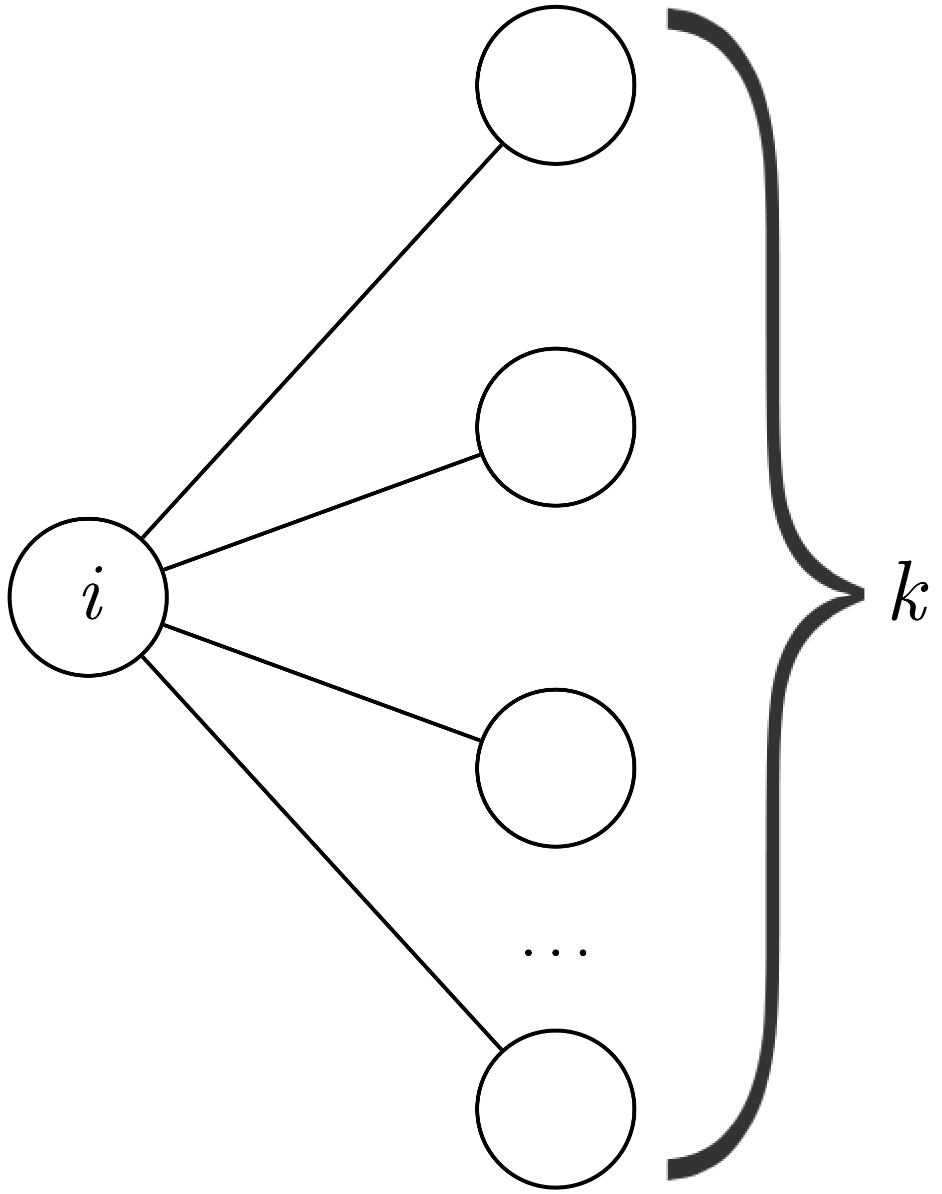

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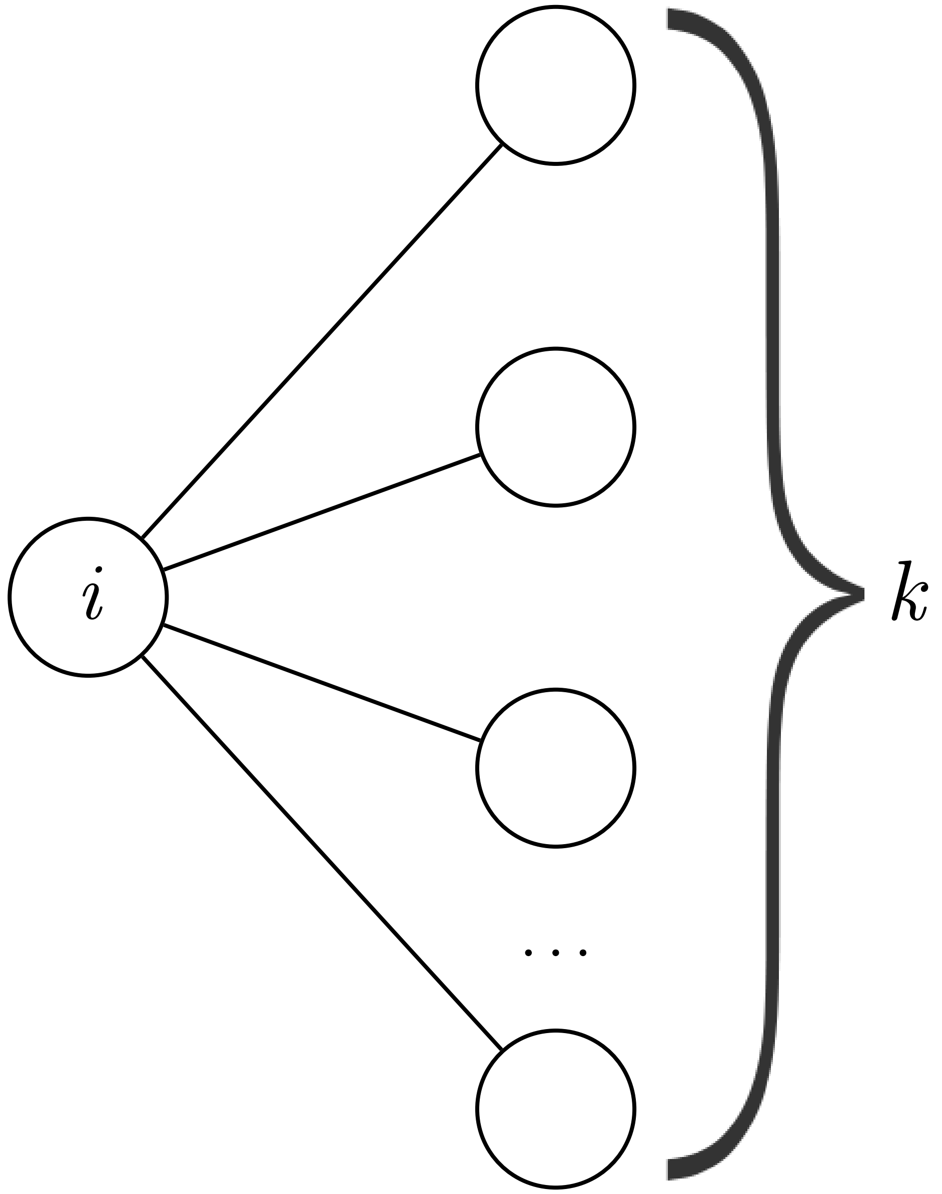
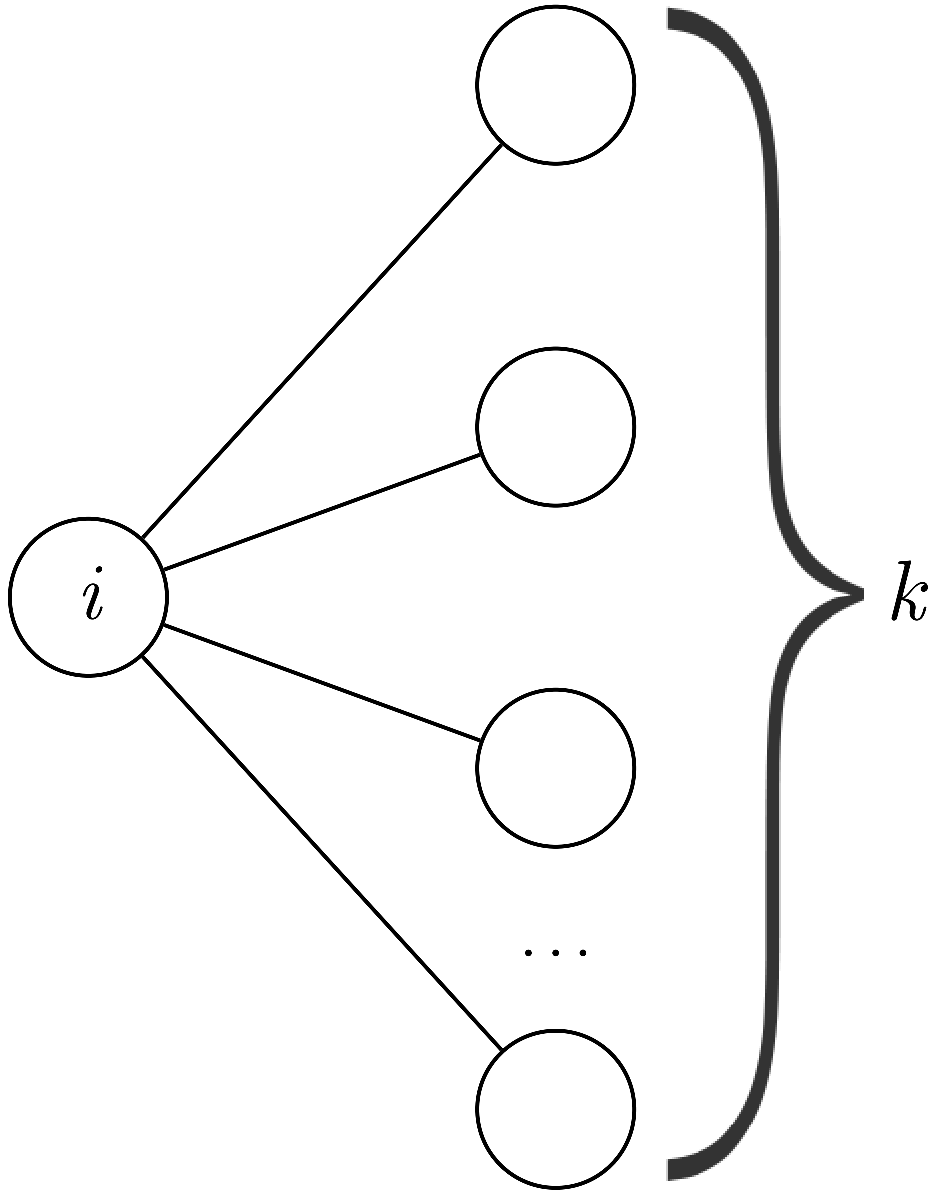
$$\Pr\left[\overline{W} > \frac{n}{2}\right] = ?$$

probability of a wrong majority after deliberation (we want it small)



Chebyshev to the rescue

$$\Pr\left[\overline{W} > \frac{n}{2}\right] = \Pr\left[\overline{W} - n \cdot m_k > n \cdot \left(\frac{1}{2} - m_k\right)\right]$$



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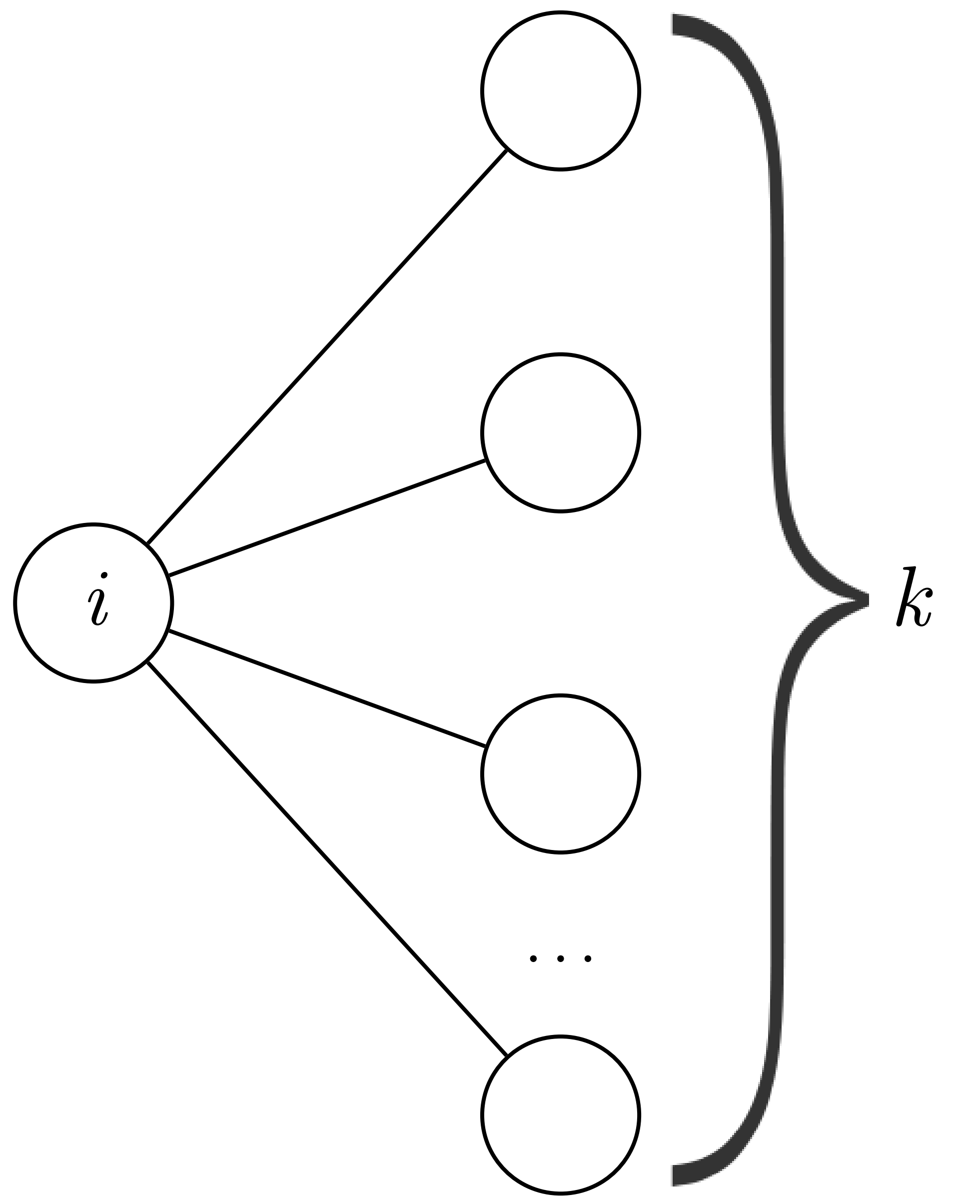

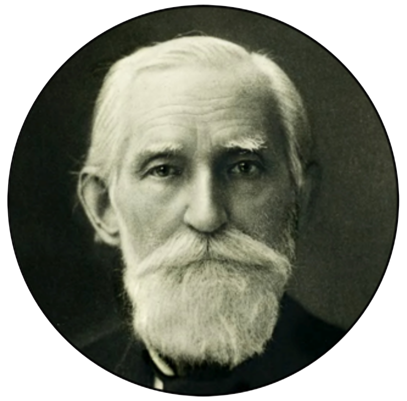

CHEBYSHEV

We need to get a handle on $\overline{W} - \mathbb{E}[\overline{W}]$.

THEOREM (CHEBYSHEV'S INEQUALITY)

If X is a random variable with finite expected value $\mathbb{E}[X]$ and variance $\mathrm{Var}[X]$, then, for any $a > 0$, it holds that:

$$\Pr\Big[|X - \mathbb{E}[X]| \geq a\Big] \leq \frac{\mathrm{Var}[X]}{a^2}.$$

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EXAMPLE

A fair coin is flipped 100 times. We want a bound on the probability that the number of heads is at least 60, or at most 40.

Take X to be the number of heads. Then, $\mathbb{E}[X] = 50$, $\mathrm{Var}[X] = 25$. And:

$$\begin{aligned} \Pr\left[X < 40, X > 60\right] &= \Pr\left[|X - \mathbb{E}[X]| \geq 10\right] \\ &\leq {}^{25}/_{10^2} \\ &= {}^{1}/_{4}. \end{aligned}$$

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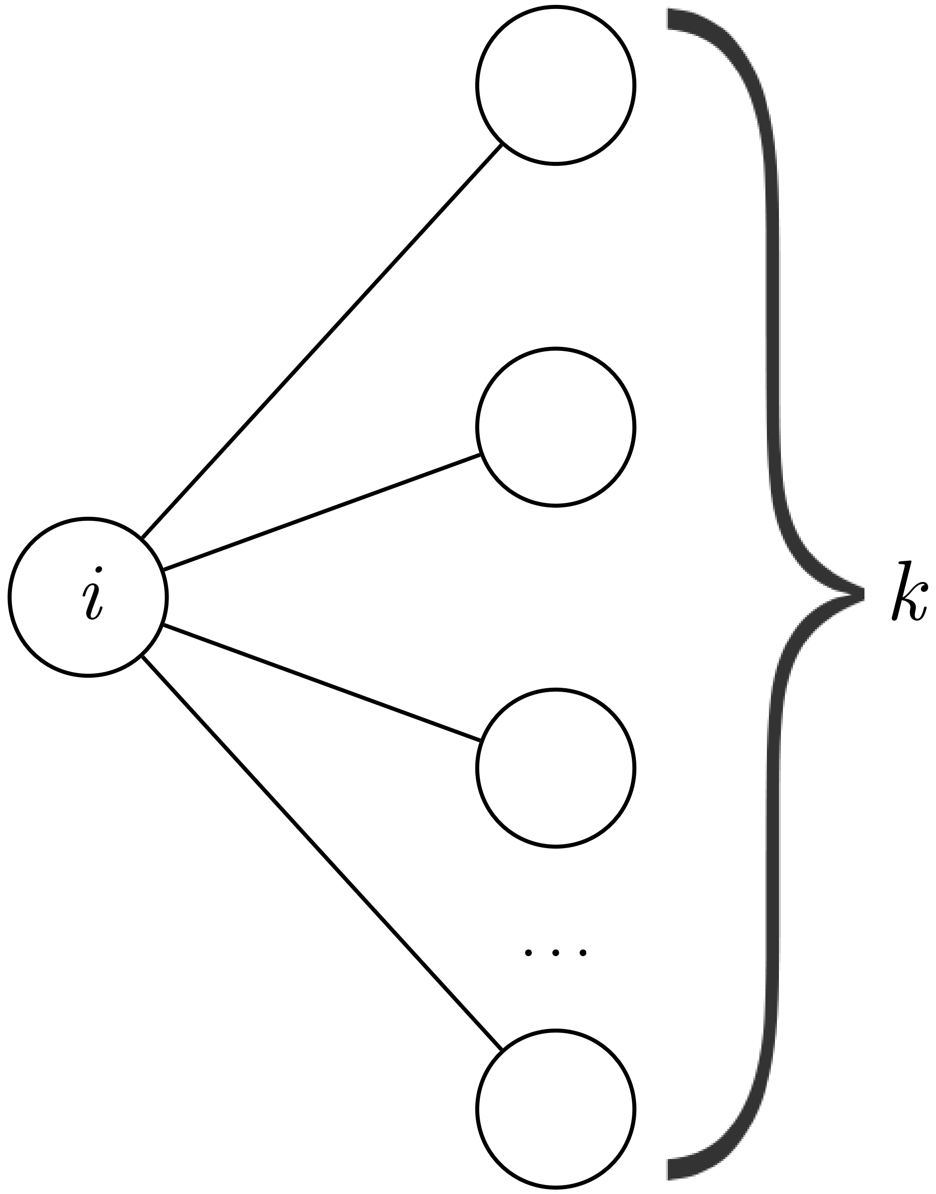

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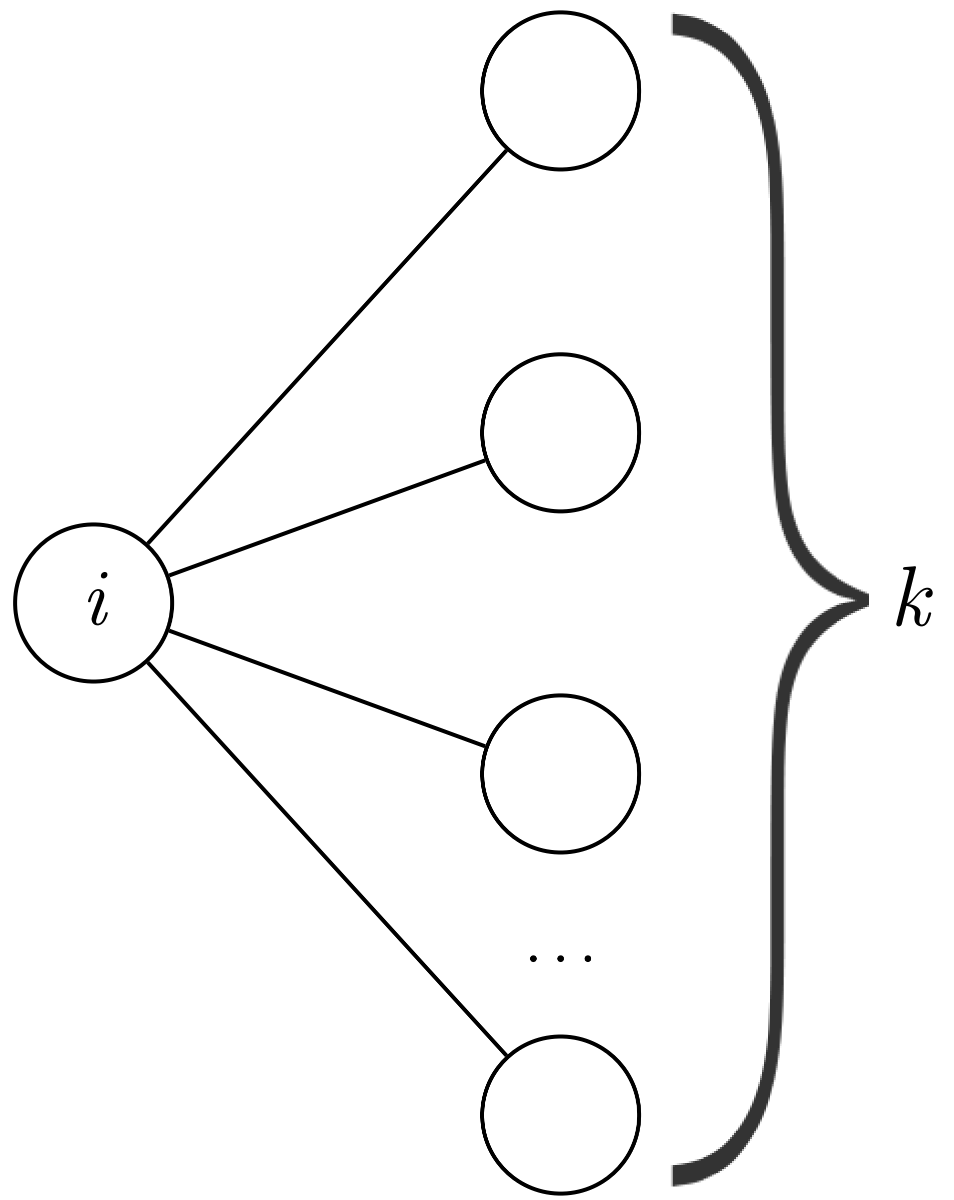

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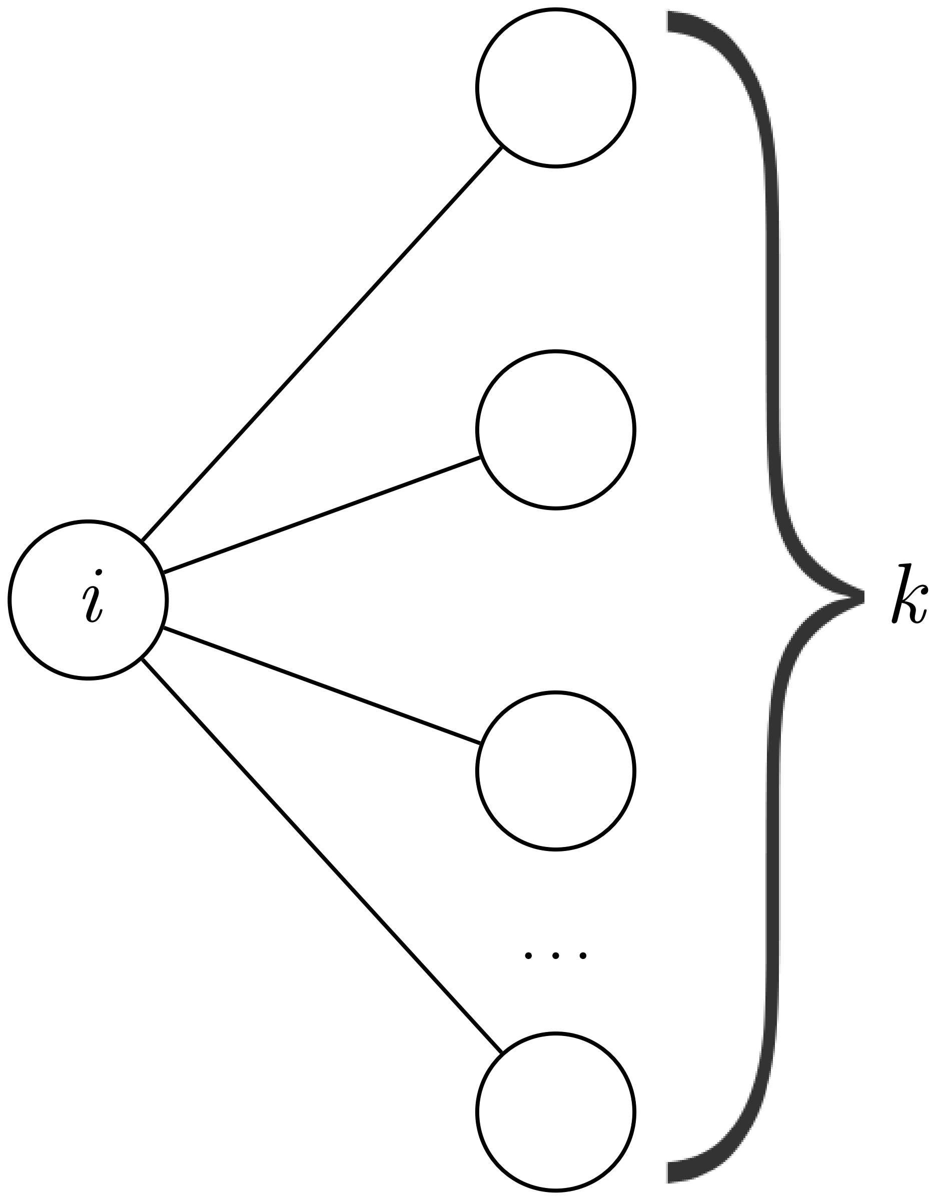
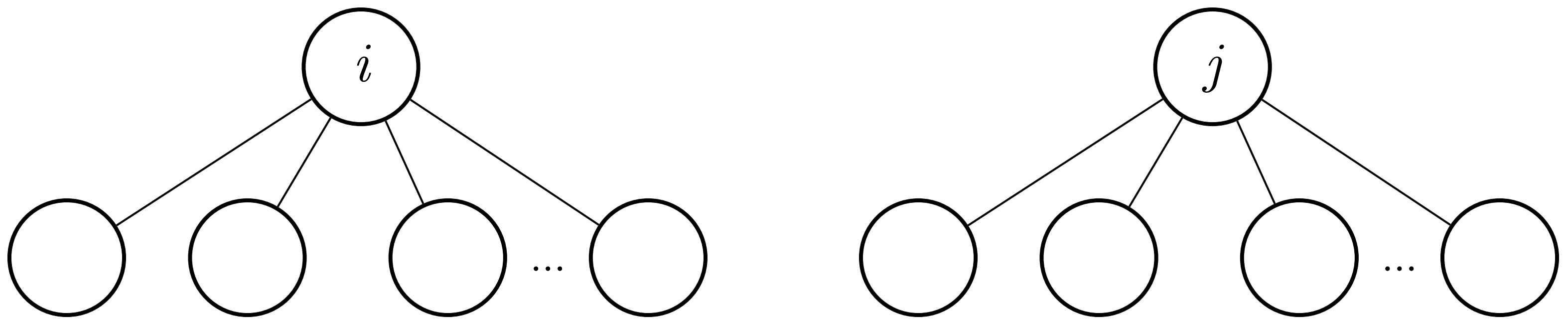
what is this?

$$= \frac{n \cdot m_k(1 - m_k) + \sum_{i \neq j} \mathrm{Cov}[W_i, W_j]}{n^2 \cdot (\frac{1}{2} - m_k)^2}$$



Figuring out the covariance

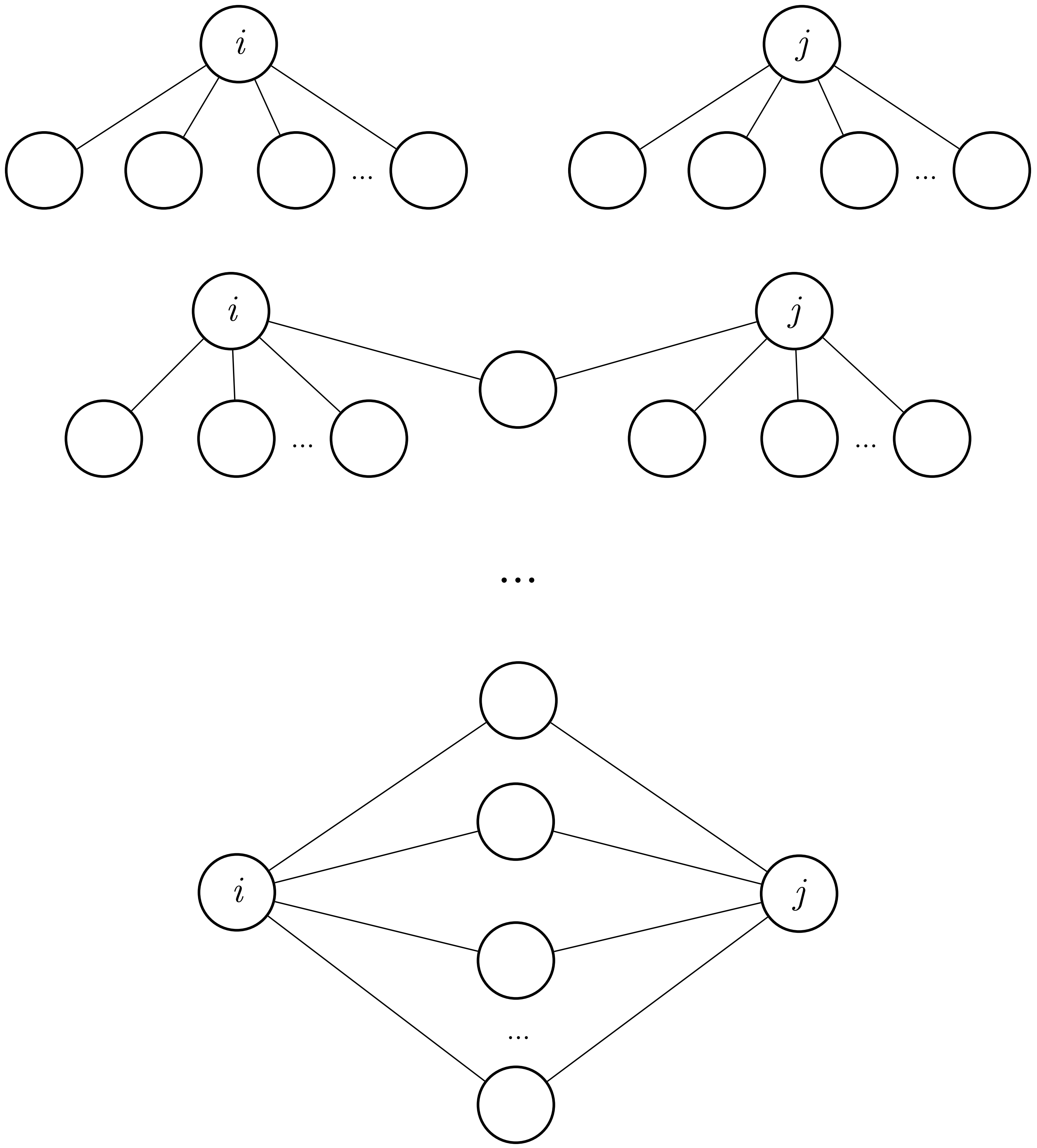


If i and j share no neighbors the covariance is 0.

Figuring out the covariance

If *i* and *j* share no neighbors the covariance is 0.

The covariance gets larger the more neighbors i and j share.



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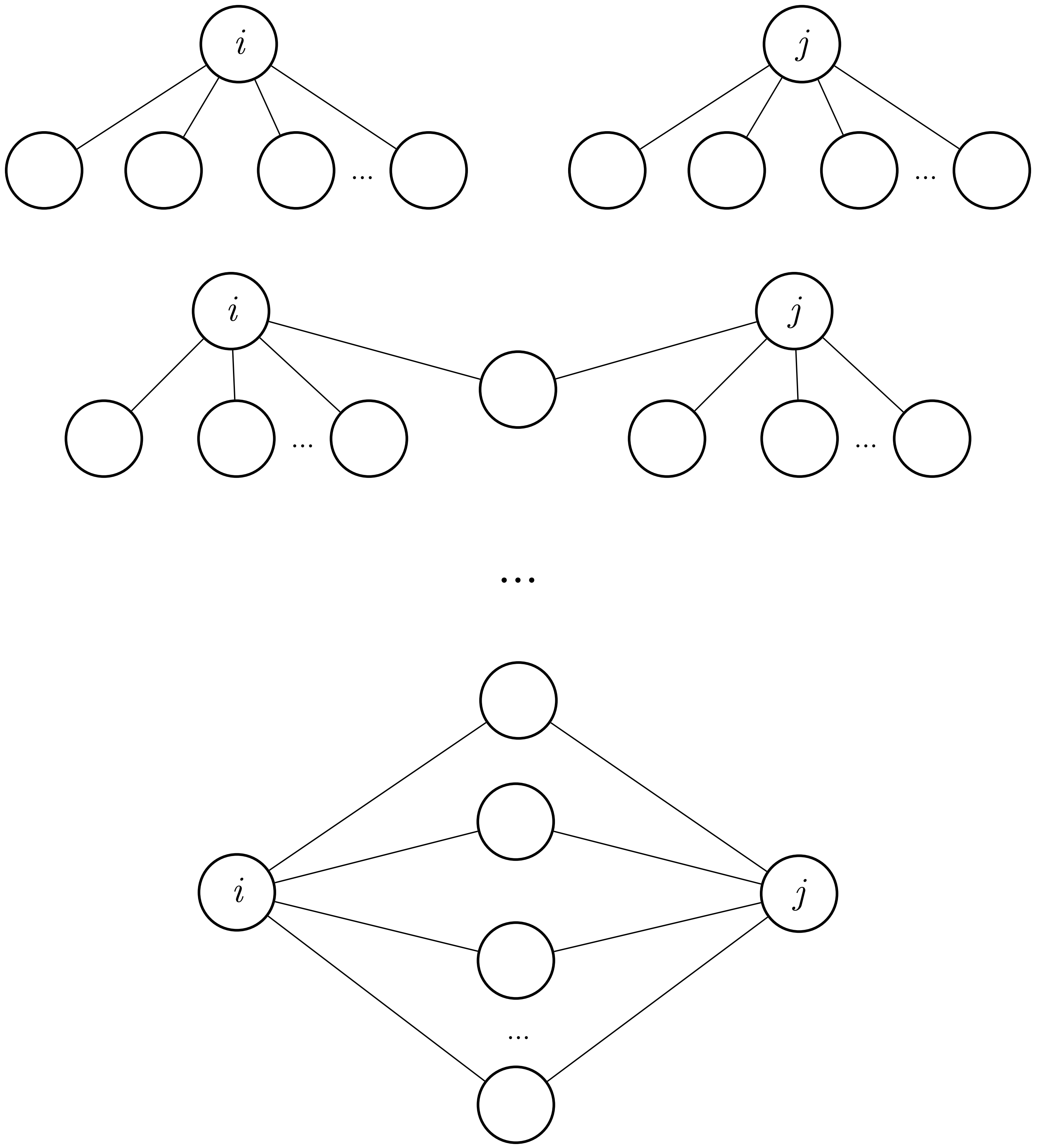

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The covariance gets larger the more neighbors i and j share.

The fraction of pairs of agents who share a neighbor goes to 0 as n goes to infinity.

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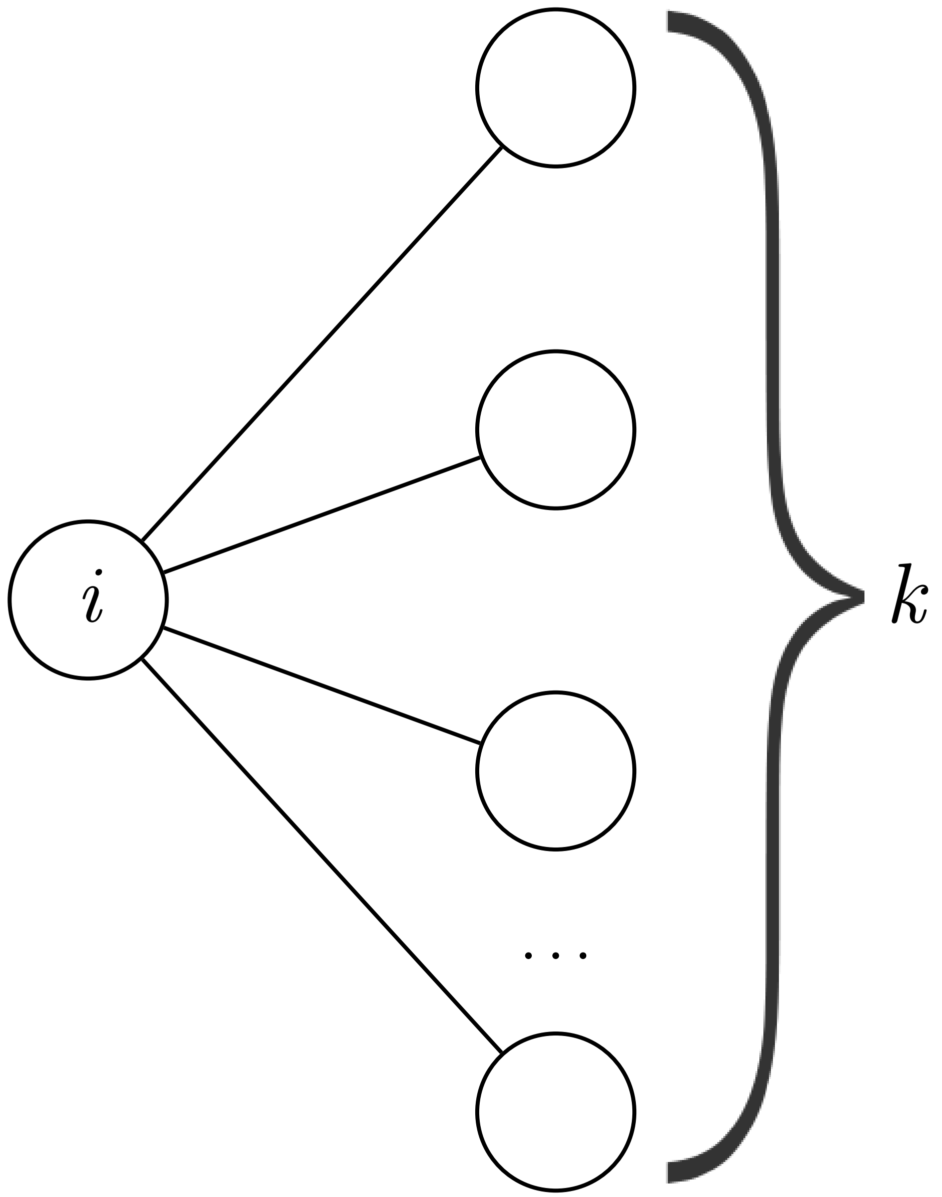
bounded



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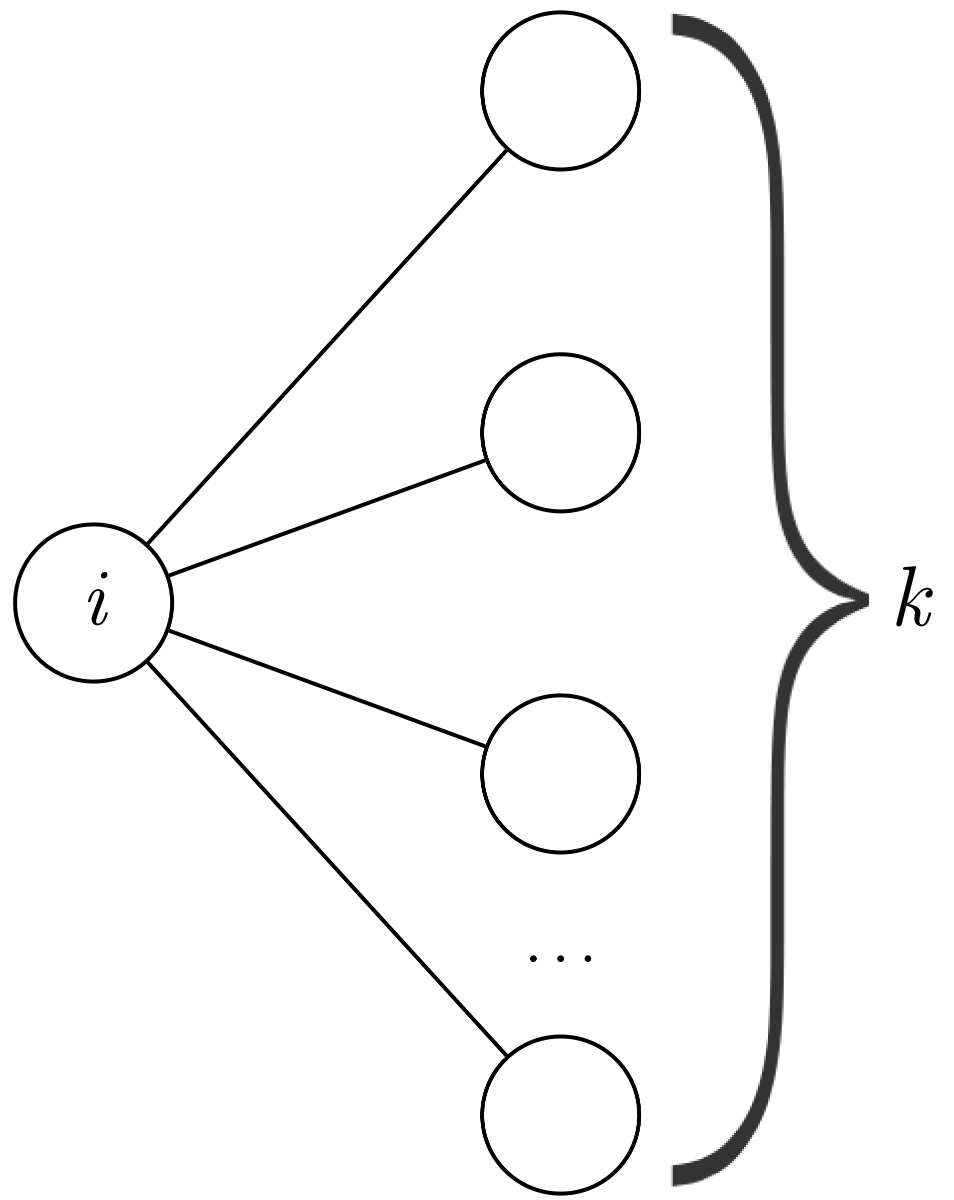
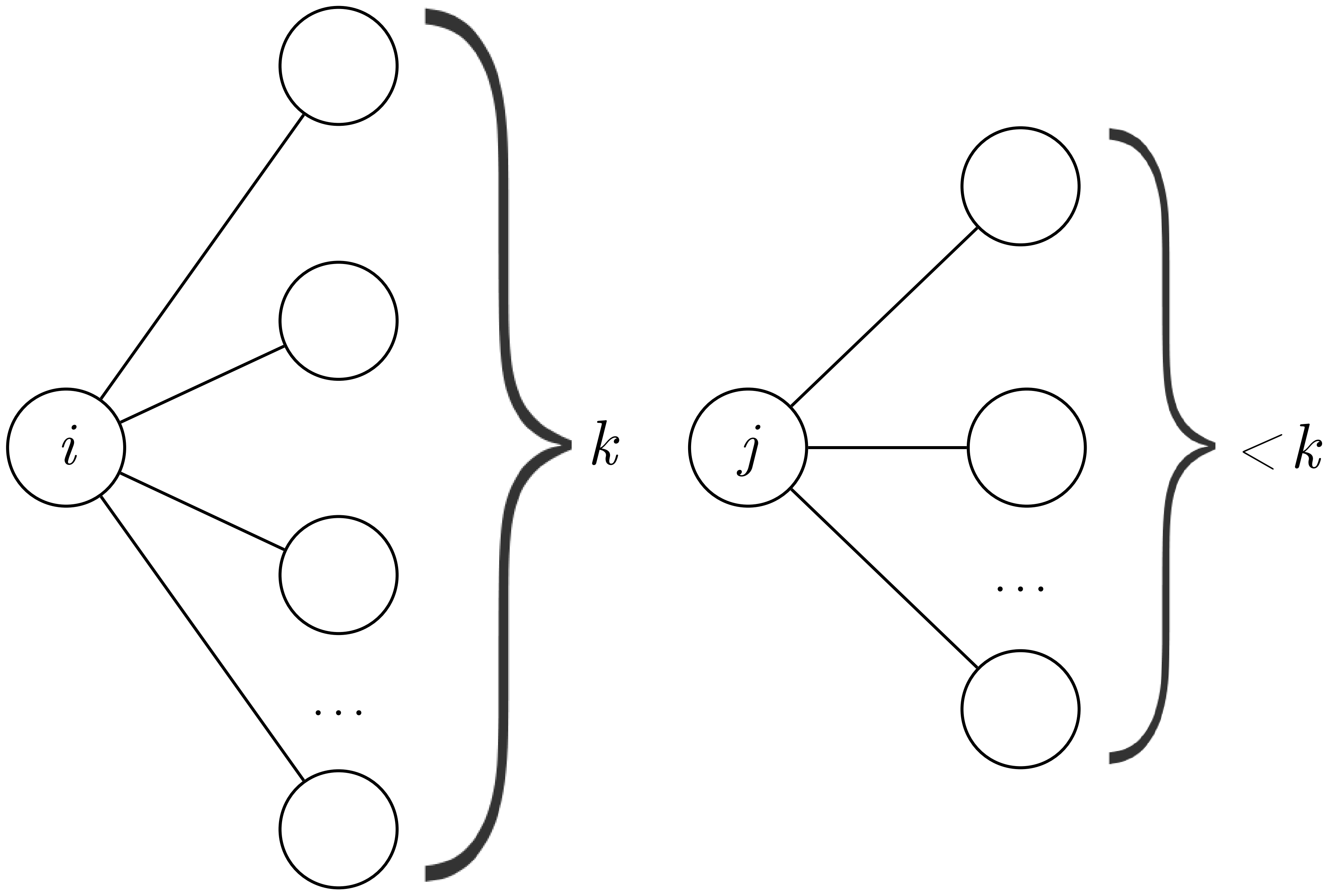
$$\to 0, \text{ as } n \to \infty$$



This can be extended to graphs where the *maximum* degree is k.

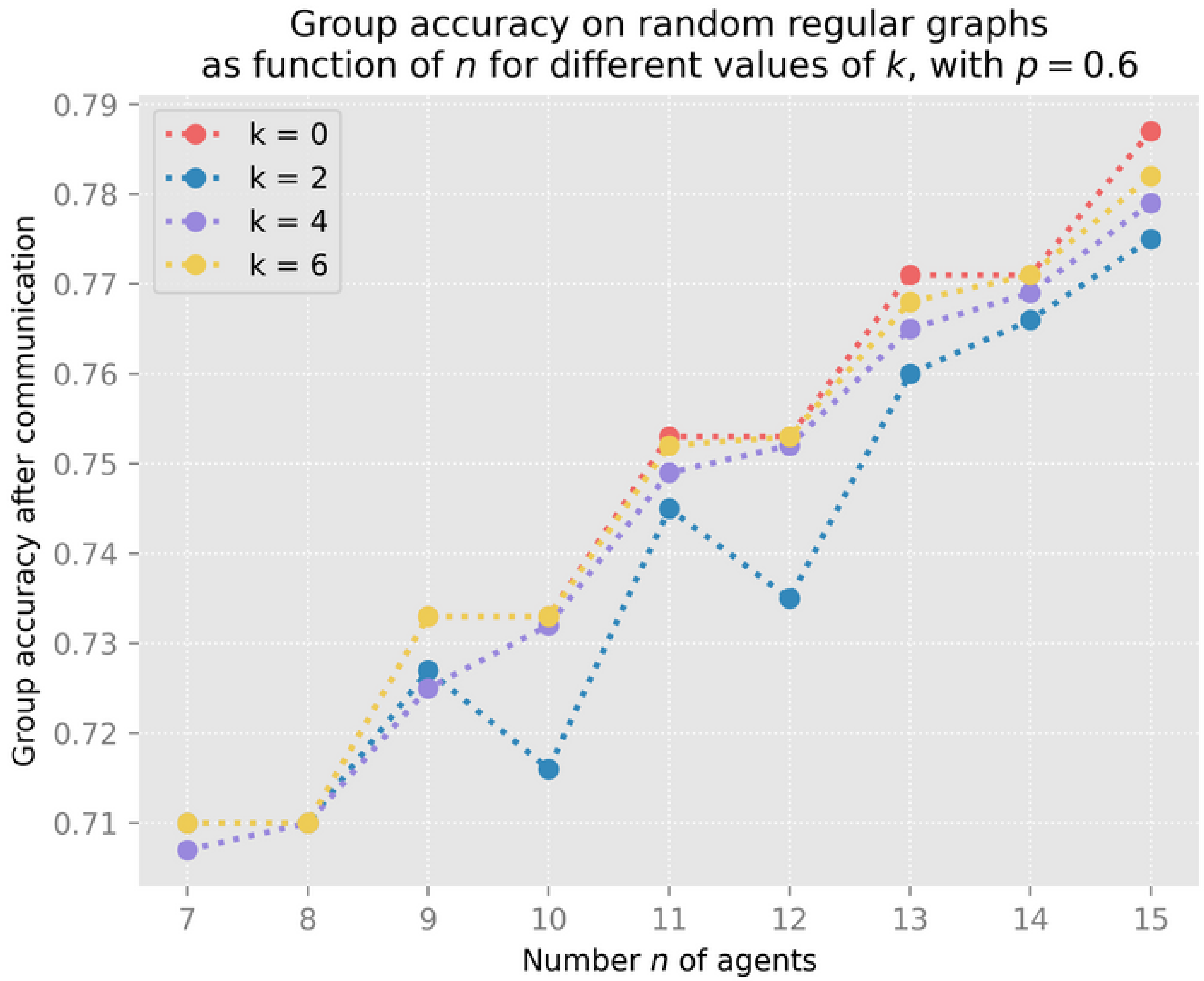


Simulation results also look promising.

Random regular graphs

On random regular graphs, accuracy grows with *k*.

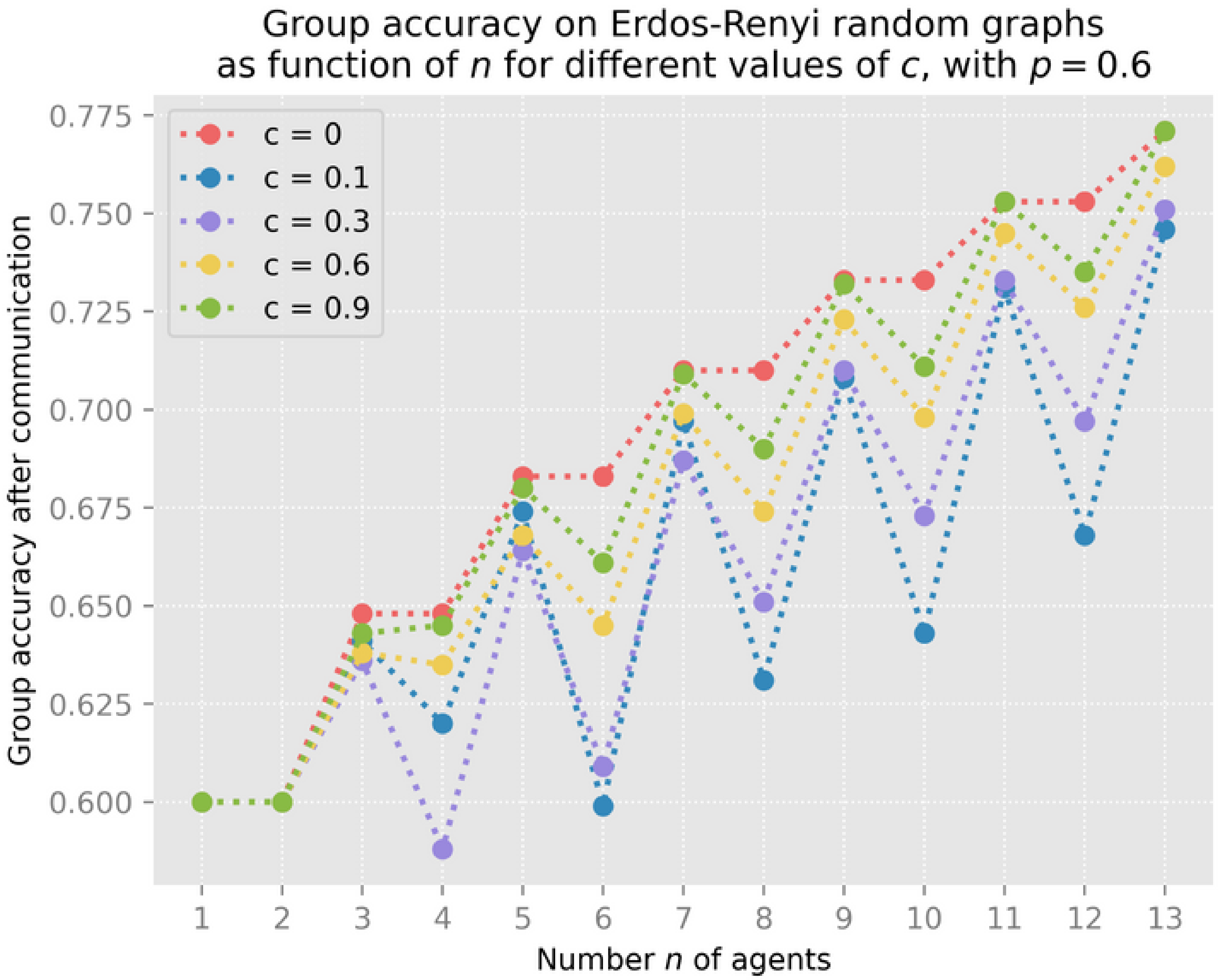
degree of every node



Erdős–Rényi random graphs

probability that any two vertices are connected

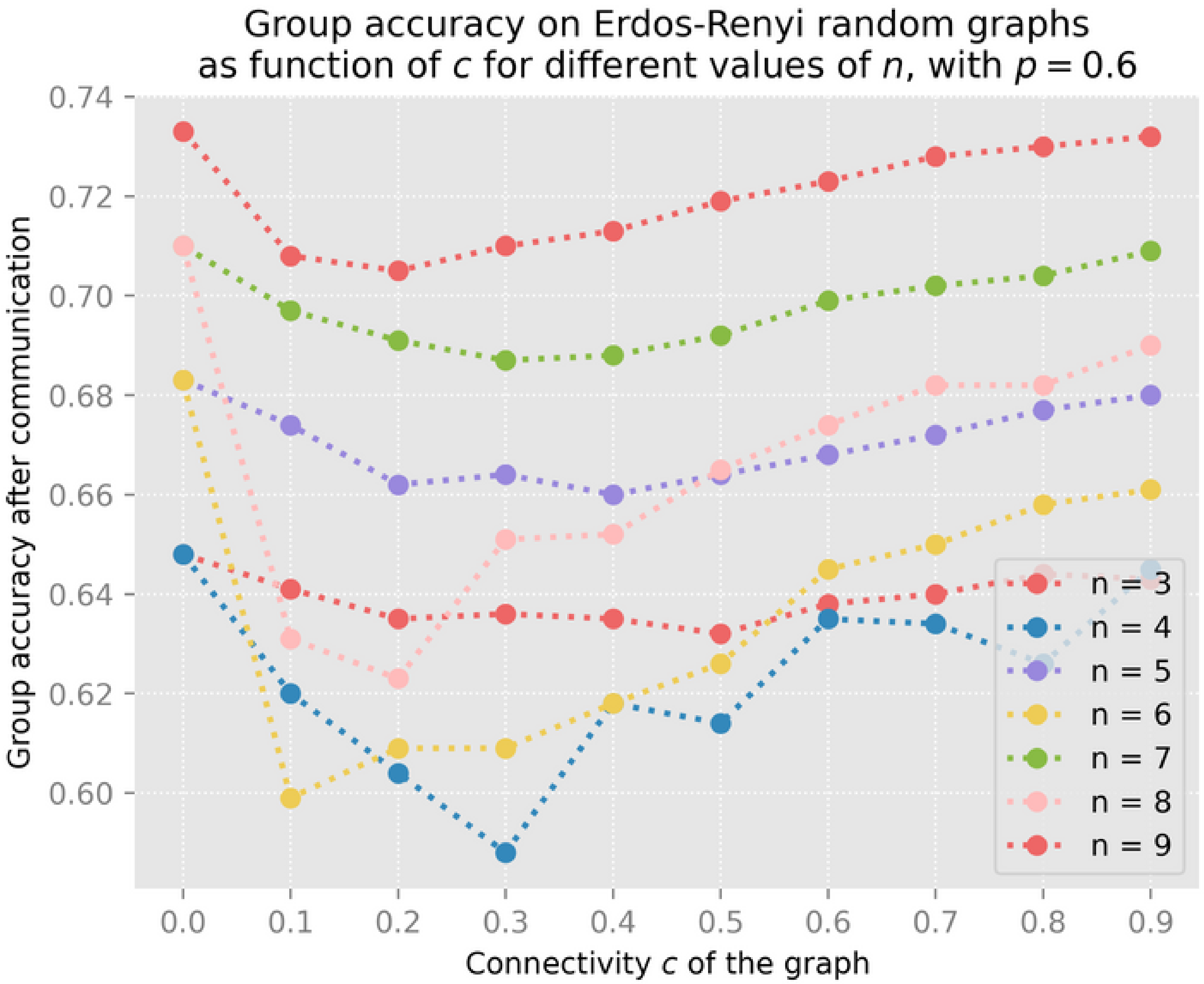
In the $G(n, c)$ model, the choice of c influences group accuracy.



An interesting thing happens on random graphs.

Erdős–Rényi random graphs

Can we be precise about the dip in accuracy?



Summing up.

NICOLIEN
For a fixed number of agents it seems that you can't do better than direct voting. :(

ADRIAN
Ideally we can bound this loss of accuracy: what's the worst it can get?

GIUSEPPE
And, optimistically, we can recover the asymptotic result for k-regular graphs.

DAVIDE
And maybe for other classes of graphs.

FREDERIK
Simulation results would give us an idea of interesting effects of the structure of the graph.